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ASYMPTOTIC SOLUTIONS OF FOURTH ORDER CRITICALLY DAMPED NONLINEAR SYSTEM UNDER SOME SPECIAL CONDITIONS

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Abstract. An asymptotic solution of a fourth order critically damped nonlinear differential system has been found by means of extended Krylov-Bogoliubov-Mitropolskii (KBM) method. The solutions obtained by this method agree with those obtained by numerical method. The method is illustrated by an example.

1. Introduction

An important approach to study nonlinear oscillatory systems is the small parameter expansion in which the perturbation theory is based. Perturbation theory comprises mathematical methods that are used to find an approximate solution to a problem. One widely used method in this theory is the averaging asymptotic method of Krylov-Bogoliubov-Mitropolskii (KBM) [2, 3]. Originally, the method was developed for obtaining periodic solutions of second order nonlinear systems with small nonlinearities. Actually, Mitropolskii was proposed a new perturbation method for investigating systems of differential equations with small perturbation term. After that Krylov and Bogoliubov agree the method of Mitropolskii but they add some conditions as the functions of the system will be continuous. So the method is called by Krylov-Bogoliubov-Mitropolskii method, that means KBM method. Asymptotic methods of non-linear mechanics developed by Krylov, Bogoliubov and Mitropolskii known as the KBM method [2] is a powerful tool for the investigation of nonlinear vibrations. Also later, the method has been extended by Popov [6] to damped oscillatory nonlinear systems. Owing to physical

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importance of this method, Mendelson [4] rediscovered the Popov's results. Murty et al. [5] also extended the KBM method for obtaining second and fourth order over-damped nonlinear systems. Sattar [7] has extended the KBM method for second order critically damped nonlinear systems. Shamsul [11] presented a new asymptotic technique for second order over-damped and critically damped nonlinear systems. In article [10], Shamsul has generalized the KBM asymptotic method. Shamsul and Sattar [8] have presented an asymptotic method for third order critically damped nonlinear equations. Shamsul [13] again found an asymptotic solution for a third order critically damped nonlinear system. In the present article, a fourth order critically damped nonlinear system is considered and desired solutions are found under some special conditions.

2. The method

Consider a fourth order weakly nonlinear system governed by the ordinary differential equation

$$x^{(4)} + k_1 \dddot{x} + k_2 \ddot{x} + k_3 \dot{x} + k_4 x = -\varepsilon f(x, \dot{x}, \ddot{x}, \dddot{x}) \tag{2.1}$$

where $x^{(4)}$ denote the fourth derivative of x , over dots are used for first, second and third derivatives with respect to t, k_1, k_2, k_3, k_4 are constants, ε is the small parameter and f is the given nonlinear function. Here $-\lambda_1, -\lambda_2, -\lambda_3$ and $-\lambda_4$ are the real negative eigen values when $\varepsilon = 0$ and two of the eigen values say λ_3 and λ_4 are equal. In this case the solution of the linear equation of (2.1) is

$$X(t, 0) = a_{1,0} e^{-\lambda_1 t} + a_{2,0} e^{-\lambda_2 t} + (a_{3,0} + a_{4,0} t) e^{-\lambda_3 t} \tag{2.2}$$

where $a_{j,0}, j = 1, 2, 3, 4$ are constants of integration.

When $\varepsilon \neq 0$ following [2, 3] a solution of the equation (2.1) is sought in the form

$$\begin{aligned} X(t, \varepsilon) = {} & a_1(t) e^{-\lambda_1 t} + a_2(t) e^{-\lambda_2 t} + (a_3(t) + t a_4(t)) e^{-\lambda_3 t} \\ & + \varepsilon u_1(a_1, a_2, a_3, a_4, t) + K \end{aligned} \tag{2.3}$$

where each $a_j, j = 1, 2, 3, 4$ each satisfy the first order differential equation

$$\dot{a_j} = \varepsilon A_j(a_1, a_2, a_3, a_4, t) + K \tag{2.4}$$

Confining only a first few terms $1, 2, 3, ..., m$ in the series expansion of (2.3) and (2.4), we evaluate the functions $u_j,\ A_j,\ j = 1, 2, 3, K, m$ such that $a_j(t)$, $j = 1, 2, 3, K, m$, appearing in (2.3) and (2.4) satisfy the given differential equation (2.1) with an accuracy of order $\varepsilon^{(m+1)}$. In order to determine these unknown functions it is customary in KBM method that the correction terms, u_j must exclude terms (known as secular terms) which make them large. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a lower order.

Now differentiating the equation (2.3) four times with respect t, substituting the value of x and the derivatives $\dot{x}, \ddot{x}, \dddot{x}, x^{(4)}$ in the original equation (2.1), utilizing the relation presented in (2.4) and finally equating the coefficients of ε , we obtain

$$\begin{aligned}
&e^{-\lambda_1 t}(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2)(\frac{\partial}{\partial t} - \lambda_1 + \lambda_3)^2 A_1 \\
&+ e^{-\lambda_2 t}(\frac{\partial}{\partial t} - \lambda_2 + \lambda_1)(\frac{\partial}{\partial t} - \lambda_2 + \lambda_3)^2 A_2 \\
&+ e^{-\lambda_3 t}(\frac{\partial}{\partial t} - \lambda_3 + \lambda_1)(\frac{\partial}{\partial t} - \lambda_3 + \lambda_2)(\frac{\partial A_3}{\partial t} + 2A_4 + t\frac{\partial A_4}{\partial t}) \\
&+ (\frac{\partial}{\partial t} + \lambda_1)(\frac{\partial}{\partial t} + \lambda_2)(\frac{\partial}{\partial t} + \lambda_3)^2 u_1 = -f^{(0)}(a_1, a_2, a_3, a_4, t)
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
&f^{(0)}(a_1, a_2, a_3, a_4, t) = f(x_0, \dot{x_0}, \ddot{x_0}, \dddot{x_0}) \quad \textit{and} \\
&x_0 = a_{1,0}e^{-\lambda_1 t} + a_{2,0}e^{-\lambda_2 t} + (a_{3,0} + a_{4,0}t)e^{-\lambda_3 t}.
\end{aligned}$$

Here, we assume that $f^{(0)}$ can be expanded in a Taylor's series as

$$\begin{aligned}
f^{(0)} &= F_0(a_1, a_2, t) + F_1(a_1, a_2, t)(a_3 + a_4 t) \\
&\quad + F_1(a_1, a_2, t)(a_3 + a_4 t)^2 + K
\end{aligned} \tag{2.6}$$

Where F_0, F_1, F_2, K do not contain the terms involving t, t^2, K . Here, first we impose the restriction that u_1 can not contain the terms with $(a_3 + a_4 t)^0$, and $(a_3 + a_4 t)^1$ of $f^{(0)}$, since these are already included in the series expansion (2.3) of order ε^0 (see also [7, 8, 11, 13, 14] for details).

The coefficients of $(a_3 + a_4 t)^n$, $n = 0, 1, 2, K$ in the equation (2.6) can be expanded in powers of $e^{-\lambda_1 t}$, $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$ of the form

$$F_0(a_1,a_2,t) = \sum_{j,k} F_{0,j,k}(a_1,a_2)e^{-(j\lambda_1+k\lambda_2)t},$$
$$F_1(a_1,a_2,t) = \sum_{j,k} F_{1,j,k}(a_1,a_2)e^{-(j\lambda_1+k\lambda_2+\lambda_3)t}, \ K \ etc. \tag{2.7}$$

Substituting the value of f^0 from equation (2.6) into equation (2.5) and equating the various power of t, we obtain

$$\begin{aligned} & e^{-\lambda_1 t}(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2)(\frac{\partial}{\partial t} - \lambda_1 + \lambda_3)^2 A_1 \\ & + e^{-\lambda_2 t}(\frac{\partial}{\partial t} - \lambda_2 + \lambda_1)(\frac{\partial}{\partial t} - \lambda_2 + \lambda_3)^2 A_2 \\ & + e^{-\lambda_3 t}(\frac{\partial}{\partial t} - \lambda_3 + \lambda_1)(\frac{\partial}{\partial t} - \lambda_3 + \lambda_2)(\frac{\partial A_3}{\partial t} + 2A_4) \\ & = -F_0(a_1,a_2,t) - a_3 F_1(a_1,a_2,t) \end{aligned} \tag{2.8}$$

$$e^{-\lambda_3 t}(\frac{\partial}{\partial t} - \lambda_3 + \lambda_1)(\frac{\partial}{\partial t} - \lambda_3 + \lambda_2)(\frac{\partial A_4}{\partial t}) = -a_4 F_1(a_1,a_2,t) \tag{2.9}$$

and

$$\begin{aligned} & (\frac{\partial}{\partial t} + \lambda_1)(\frac{\partial}{\partial t} + \lambda_2)(\frac{\partial}{\partial t} + \lambda_3)^2 u_1 \\ & \qquad = -F_2(a_1,a_2,t)(a_3 + a_4 t)^2 + K. \end{aligned} \tag{2.10}$$

Now substituting the value of $F_0(a_1,a_2,t)$ from equation (2.7) into equation (2.9), we obtain

$$\begin{aligned} & A_4 = \\ & \sum_{j,k} \frac{a_4 F_{1,j,k}(a_1,a_2)e^{-(j\lambda_1+k\lambda_2)t}}{(j\lambda_1+k\lambda_2)(j\lambda_1+k\lambda_2-\lambda_1+\lambda_3)(j\lambda_1+k\lambda_2-\lambda_2+\lambda_3)} \end{aligned} \tag{2.11}$$

Finally, substituting the value of $F_0(a_1,a_2 t)$ and $F_1(a_1,a_2 t)$ from the equation (2.7) and the value of A_4 from equation (2.11) into equation (2.8), we obtain

$$
\begin{aligned}
&e^{-\lambda_1 t}(\frac{\partial}{\partial t}-\lambda_1+\lambda_2)(\frac{\partial}{\partial t}-\lambda_1+\lambda_3)^2 A_1\\
&+e^{-\lambda_2 t}(\frac{\partial}{\partial t}-\lambda_2+\lambda_1)(\frac{\partial}{\partial t}-\lambda_2+\lambda_3)^2 A_2\\
&+e^{-\lambda_3 t}(\frac{\partial}{\partial t}-\lambda_3+\lambda_1)(\frac{\partial}{\partial t}-\lambda_3+\lambda_2)(\frac{\partial A_3}{\partial t})\\
&=-\sum_{j,k} F_{0,j,k}(a_1,a_2)e^{-(j\lambda_1+k\lambda_2)t}\\
&-\sum_{j,k} a_3 F_{1,j,k}(a_1,a_2)e^{-(j\lambda_1+k\lambda_2+\lambda_3)t}\\
&-2\sum_{j,k}\frac{a_4 F_{1,j,k}(a_1,a_2)e^{-(j\lambda_1+k\lambda_2+\lambda_3)t}}{(j\lambda_1+k\lambda_2)}.
\end{aligned}
\tag{2.12}
$$

Now it is not easy to solve the equation (2.12) for the unknown functions A_1, A_2 and A_3 , if the nonlinear function f and the eigen values $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4$ of the linear equation of (2.1) are not specified. When these are specified the value of A_1, A_2 and A_3 can be found subject to the condition that the coefficient in the solution of A_1, A_2 and A_3 do not become large (see also [1, 17] for details).
Equation (2.10) is a fourth order nonhomogeneous linear differential equation. When the nonlinear function f is specified, we can find the particular solution of the equation (2.10) for the unknown function u_1. This completes the determination of the first order solution of the equation (2.1).

3. Example

As an example of the above method, we consider the Duffing equation type nonlinear system

$$
x^{(4)}+k_1\dddot{x}+k_2\ddot{x}+k_3\dot{x}+k_4 x=-\varepsilon f(x,\dot{x},\ddot{x},\dddot{x}). \tag{3.1}
$$

Here $f=x^3$.

Therefore

$$
\begin{aligned}
f^{(0)} &= {a_1}^3 e^{-3\lambda_1 t} + 3{a_1}^2 a_2 e^{-(2\lambda_1+\lambda_2)t} \\
&+ 3a_1 a_2^2 e^{-(\lambda_1+2\lambda_2)t} + a_2^3 e^{-3(\lambda_2 t)} \\
&+ 3({a_1}^2 e^{-(2\lambda_1+\lambda_3)t} + 2a_1 a_2 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + {a_2}^2 e^{-(2\lambda_2+\lambda_3)t}) \\
&\qquad\qquad\qquad\qquad \times (a_3 + a_4 t) \\
&+ 3(a_1 e^{-(\lambda_1+2\lambda_3)t} + a_2 e^{-(\lambda_2+2\lambda_3)t})(a_3+a_4)^2 \\
&+ e^{-3\lambda_3 t}(a_3+a_4 t)^3
\end{aligned} \tag{3.2}
$$

Therefore, we have

$$
\begin{aligned}
& e^{-\lambda_1 t}\Big(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\Big)\Big(\frac{\partial}{\partial t} - \lambda_1 + \lambda_3\Big)^2 A_1 \\
& + e^{-\lambda_2 t}\Big(\frac{\partial}{\partial t} - \lambda_2 + \lambda_1\Big)\Big(\frac{\partial}{\partial t} - \lambda_2 + \lambda_3\Big)^2 A_2 \\
& e^{-\lambda_3 t}\Big(\frac{\partial}{\partial t} - \lambda_3 + \lambda_1\Big)\Big(\frac{\partial}{\partial t} - \lambda_3 + \lambda_2\Big)\Big(\frac{\partial A_3}{\partial t} + 2A_4\Big) \\
& = -\Big\{{a_1}^3 e^{-3\lambda_1 t} + 3{a_1}^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + 3a_1 {a_2}^2 e^{-(\lambda_1+2\lambda_2)t} \\
& + {a_2}^3 e^{-3\lambda_2 t} + 3{a_1}^2 a_3 e^{-(2\lambda_1+\lambda_3)t} + 6a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} \\
& + 3{a_2}^2 a_3 e^{-(2\lambda_2+\lambda_3)t}\Big\}
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
& e^{-\lambda_3 t}\Big(\frac{\partial}{\partial t} - \lambda_3 + \lambda_1\Big)\Big(\frac{\partial}{\partial t} - \lambda_3 + \lambda_2\Big)\Big(\frac{\partial A_4}{\partial t}\Big) \\
& \qquad = -3\Big\{{a_1}^2 a_4 e^{-(2\lambda_1+\lambda_3)t} + 2a_1 a_2 a_4 e^{-(\lambda_1+\lambda_2+\lambda_3)t} \\
& \qquad + {a_2}^2 a_4 e^{-(2\lambda_2+\lambda_3)t}\Big\}
\end{aligned} \tag{3.4}
$$

and

(3.5)

$$
\begin{aligned}
& \Big(\frac{\partial}{\partial t} + \lambda_1\Big)\Big(\frac{\partial}{\partial t} + \lambda_2\Big)\Big(\frac{\partial}{\partial t} + \lambda_3\Big)^2 u_1 \\
& = -\Big\{3(a_1 {a_3}^2 e^{-(\lambda_1+2\lambda_3)t} + {a_3}^2 a_2 e^{-(2\lambda_3+\lambda_2)t}) + {a_3}^3 e^{-3\lambda_3 t} \\
& + (6a_1 a_3 a_4 e^{-(\lambda_1+2\lambda_3)t} + 6a_2 a_3 a_4 e^{-(\lambda_2+2\lambda_3)t} + 3{a_3}^2 a_4 e^{-3\lambda_3 t})t \\
& + 3(a_1 {a_4}^2 e^{-(\lambda_1+2\lambda_3)t} + a_2 {a_4}^2 e^{-(2\lambda_3+\lambda_2)t} + a_3 {a_4}^2 e^{-3\lambda_3 t})t^2 + {a_4}^3 e^{-3\lambda_3 t} t^3\Big\}
\end{aligned}
$$

Consequently, solving the equations (3.4) and (3.5), we obtain

$$A_4 = r_1{a_1}^2 a_4 e^{-2\lambda_1 t} + r_2 a_1 a_2 a_4 e^{-(\lambda_1+\lambda_2)t} + r_3 a_4 {a_2}^2 e^{-2\lambda_2 t}. \tag{3.6}$$

Here

$$r_1 = \frac{3}{2\lambda_1(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)}, r_2 = \frac{6}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_1+\lambda_3)},$$
$$r_3 = \frac{3}{2\lambda_2(\lambda_2+\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)}$$

and

$$\begin{aligned} u_1 =& m_1 a_1 {a_3}^2 e^{-(\lambda_1+2\lambda_3)t} + m_2 a_2 {a_3}^2 e^{-(\lambda_1+2\lambda_3)t} \\ &+ m_3 {a_3}^3 e^{-3\lambda_3 t} + a_1 a_3 a_4 e^{-(\lambda_1+2\lambda_3)t}(2m_1 t + m_4) \\ &+ a_2 a_3 a_4 e^{-(\lambda_1+2\lambda_3)t}(2m_2 t + m_5) \\ &+ {a_3}^2 a_4 e^{-(\lambda_1+2\lambda_3)t}(3m_3 t + m_6) \\ &+ a_1 {a_4}^2 e^{-(\lambda_1+2\lambda_3)t}(m_1 t^2 + m_7 t + m_8) \\ &+ a_2 {a_4}^2 e^{-(\lambda_1+2\lambda_3)t}(m_2 t^2 + m_9 t + m_{10}) \\ &+ a_3 {a_4}^2 e^{-3\lambda_3 t}(3m_3 t^2 + m_{11} t + m_{12}) \\ &+ {a_4}^3 e^{-3\lambda_3 t}(m_3 t^3 + m_{13} t^2 + m_{14} t + m_{15}), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} m_1 &= -\frac{3}{\{2\lambda_3(\lambda_1+2\lambda_3-\lambda_2)(\lambda_1+\lambda_3)^2\}}, \\ m_2 &= -\frac{3}{\{2\lambda_3(\lambda_2+2\lambda_3-\lambda_1)(\lambda_2+\lambda_3)^2\}}, \\ m_3 &= -\frac{1}{\{4\lambda_3^2(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)\}}, \\ m_4 &= 2m_1\Big(\frac{2}{(\lambda_1+\lambda_3)} + \frac{1}{(\lambda_1+2\lambda_3-\lambda_2)} + \frac{1}{2\lambda_3}\Big), \\ m_5 &= 2m_2\Big(\frac{2}{(\lambda_2+\lambda_3)} + \frac{1}{(\lambda_2+2\lambda_3-\lambda_1)} + \frac{1}{2\lambda_3}\Big), \\ m_6 &= 3m_3\Big(\frac{2}{(3\lambda_3-\lambda_1)} + \frac{1}{(3\lambda_3-\lambda_2)} + \frac{1}{\lambda_3}\Big), \\ m_7 &= m_1\Big(\frac{4}{(\lambda_1+\lambda_3)} + \frac{2}{(\lambda_1+2\lambda_3-\lambda_2)} + \frac{1}{\lambda_3}\Big), \end{aligned}$$

$$
\begin{aligned}
m_8 = m_1\Big(&\frac{4}{(\lambda_1+\lambda_3)^2} + \frac{6}{(\lambda_1+\lambda_3)^2} + \frac{2}{(\lambda_1+2\lambda_3-\lambda_2)^2}\\
&+ \frac{2}{\lambda_3(\lambda_1+\lambda_3)} + \frac{1}{\lambda_3(\lambda_1-2\lambda_3-\lambda_2)} + \frac{1}{2{\lambda_3}^2}\Big),\\
m_9 = m_2\Big(&\frac{4}{(\lambda_2+\lambda_3)} + \frac{2}{(\lambda_2+2\lambda_3-\lambda_1)} + \frac{1}{\lambda_3}\Big),\\
m_{10} = m_2\Big(&\frac{4}{(\lambda_2+\lambda_3)} + \frac{6}{(\lambda_2+\lambda_3)^2} + \frac{2}{(\lambda_2+2\lambda_3-\lambda_1)^2}\\
&+ \frac{2}{\lambda_3(\lambda_2+\lambda_3)} + \frac{1}{\lambda_3(\lambda_2+2\lambda_3-\lambda_1)} + \frac{1}{2{\lambda_3}^2}\Big),\\
m_{11} = 3m_3\Big(&\frac{2}{\lambda_3} + \frac{2}{(3\lambda_3-\lambda_1)} + \frac{2}{(3\lambda_3-\lambda_2)}\Big),\\
m_{12} = 3m_3\Big(&\frac{3}{2{\lambda_3}^2} + \frac{2}{\lambda_3(3\lambda_3-\lambda_1)} + \frac{2}{\lambda_3(3\lambda_3-\lambda_1)}\\
&+ \frac{2}{(3\lambda_3-\lambda_1)^2} + \frac{2}{(3\lambda_3-\lambda_2)^2} + \frac{2}{(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}\Big),\\
m_{13} = m_3\Big(&\frac{3}{\lambda_3} + \frac{3}{(3\lambda_3-\lambda_1)} + \frac{3}{(3\lambda_3-\lambda_2)}\Big),\\
m_{14} = m_3\Big(&\frac{9}{2{\lambda_3}^2} + \frac{6}{\lambda_3(3\lambda_3-\lambda_1)} + \frac{6}{\lambda_3(3\lambda_3-\lambda_2)}\\
&+ \frac{6}{(3\lambda_3-\lambda_1)^2} + \frac{6}{(3\lambda_3-\lambda_2)^2} + \frac{6}{(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}\Big),\\
m_{15} = m_3\Big(&\frac{3}{{\lambda_3}^3} + \frac{9}{2\lambda_3^2(3\lambda_3-\lambda_1)} + \frac{9}{2\lambda_3^2(3\lambda_3-\lambda_2)}\\
&+ \frac{6}{\lambda_3(3\lambda_3-\lambda_1)^2} + \frac{6}{\lambda_3(3\lambda_3-\lambda_2)^2} + \frac{6}{(3\lambda_3-\lambda_1)^3}\\
&+ \frac{6}{(3\lambda_3-\lambda_2)^3} + \frac{6}{(3\lambda_3-\lambda_1)^2(3\lambda_3-\lambda_2)}\\
&+ \frac{6}{(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)^2} + \frac{6}{\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}\Big).
\end{aligned}
$$

Substituting the value of A_4 from equtaion (3.6) into equtaion (3.3), we obtain

$$
\begin{aligned}
&e^{-\lambda_1 t}\Big(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\Big)\Big(\frac{\partial}{\partial t} - \lambda_1 + \lambda_3\Big)^2 A_1\\
&\quad + e^{-\lambda_2 t}\Big(\frac{\partial}{\partial t} - \lambda_2 + \lambda_1\Big)\Big(\frac{\partial}{\partial t} - \lambda_2 + \lambda_3\Big)^2 A_2
\end{aligned}
$$

$$(3.8)\quad \begin{aligned} & e^{-\lambda_3 t}\Big(\frac{\partial}{\partial t}-\lambda_3+\lambda_1\Big)\Big(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\Big)\Big(\frac{\partial A_3}{\partial t}\Big) \\ &\quad = -\Big\{a_3 e^{-3\lambda_1}t + 3{a_1}^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{-(\lambda_1+2\lambda_2)t} \\ &\quad + {a_2}^3 e^{-3\lambda_2 t} + 3{a_1}^2 a_3 e^{-(2\lambda_1+\lambda_3)t} + 6a_1a_2a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} \\ &\quad + 3a_2^2 a_3 e^{-(2\lambda_2+\lambda_3)t} + \frac{3}{\lambda_1} a_1^2 a_4 e^{-(2\lambda_1+\lambda_3)t} \\ &\quad + \frac{12}{\lambda_1+\lambda_2} a_1a_2a_4 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + \frac{3}{\lambda_2}{a_2}^2 a_4 e^{-(2\lambda_2+\lambda_3)t}\Big\}. \end{aligned}$$

To separate the equation (3.8), for determining the unknown functions A_1, A_2 and A_3, we consider the relations (see also [1,16] for details) among the eigen values as $\lambda_1 \approx 2\lambda_2, \lambda_1 \approx \lambda_2 + 2\lambda_3$ and $\lambda_3 = \lambda_4$ for critical condition. Under these conditions, we obtain

$$(3.9)\quad \begin{aligned} & e^{-\lambda_1 t}\Big(\frac{\partial}{\partial t}-\lambda_1+\lambda_2\Big)\Big(\frac{\partial}{\partial t}-\lambda_1+\lambda_3\Big)^2 A_1 \\ &\quad = -\Big\{a_3 e^{-3\lambda_1 t} + 3{a_1}^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{-(\lambda_1+2\lambda_2)t} \\ &\quad + {a_2}^3 e^{-3\lambda_2 t} + 3{a_1}^2 a_3 e^{-(2\lambda_1+\lambda_3)t} \\ &\quad + 6a_1a_2a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 3a_2^2 a_3 e^{-(2\lambda_2+\lambda_3)t} \\ &\quad + \frac{3}{\lambda_1} a_1^2 a_4 e^{-(2\lambda_1+\lambda_3)t} \\ & + \frac{12}{\lambda_1+\lambda_2} a_1a_2a_4 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + \frac{3}{\lambda_2}{a_2}^2 a_4 e^{-(2\lambda_2+\lambda_3)t}\Big\} \end{aligned}$$

$$(3.10)\quad e^{-\lambda_2 t}\Big(\frac{\partial}{\partial t}-\lambda_2+\lambda_1\Big)\Big(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\Big)^2 A_2 = 0$$

$$(3.11)\quad e^{-\lambda_3 t}\Big(\frac{\partial}{\partial t}-\lambda_3+\lambda_1\Big)\Big(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\Big)^2 \frac{\partial A_3}{\partial t} = 0.$$

The particular solutions of equations (3.9)-(3.11) are

$$(3.12)\quad \begin{aligned} A_1 &= l_1{a_1}^3 e^{-2\lambda_1 t} + l_2{a_1}^2 a_2 e^{-(\lambda_1+\lambda_2)t} + l_3 a_1{a_2}^2 e^{-2\lambda_2 t} \\ &+ l_4{a_2}^3 e^{-(3\lambda_2-\lambda_1)t} + l_5{a_1}^2 a_3 e^{-(\lambda_1+\lambda_3)t} + l_6 a_1a_2a_3 e^{-(\lambda_2+\lambda_3)t} \\ &+ l_7{a_2}^2 a_3 e^{-(2\lambda_2+\lambda_3-\lambda_1)t} + l_8{a_1}^2 a_4 e^{-(\lambda_1+\lambda_3)t} \\ &+ l_9 a_1a_2a_4 e^{-(\lambda_2+\lambda_3)t} + l_{10}{a_2}^2 a_4 e^{-(2\lambda_2+\lambda_3-\lambda_1)t} \end{aligned}$$

$$A_2 = 0$$
$$A_3 = 0$$

where

$$
\begin{aligned}
l_1 &= \frac{1}{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)^2}, \quad & l_2 &= \frac{3}{2\lambda_1(2\lambda_1+\lambda_2-\lambda_3)^2},\\
l_3 &= \frac{3}{(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)^2}, \quad & l_4 &= \frac{1}{2\lambda_2(3\lambda_2-\lambda_3)^2}\\
l_5 &= \frac{3}{4{\lambda_1}^2(2\lambda_1-\lambda_2+\lambda_3)}, \quad & l_6 &= \frac{6}{(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)}\\
l_7 &= \frac{3}{4{\lambda_2}^2(\lambda_2+\lambda_3)}, \quad & l_8 &= \frac{3}{4\lambda_1^3(2\lambda_1-\lambda_2+\lambda_3)}\\
l_9 &= \frac{12}{(\lambda_1+\lambda_2)^3(\lambda_1+\lambda_3)}, \quad & l_{10} &= \frac{3}{4{\lambda_2}^3(\lambda_2+\lambda_3)}.
\end{aligned} \tag{3.13}
$$

Substituting the values of A_1, A_2, A_3 and A_4 from equations (3.6) and (3.12) into (2.4), we obtain

$$
\begin{aligned}
\dot{a_1} = \varepsilon\{& l_1{a_1}^3e^{-2\lambda_1 t} + l_2{a_1}^2a_2e^{-(\lambda_1+\lambda_2)t} + l_3a_1{a_2}^2e^{-2\lambda_2 t}\\
&+ l_4{a_2}^3e^{-(3\lambda_2-\lambda_1)t} + l_5{a_1}^2a_3e^{-(\lambda_1-\lambda_3)t} + l_6a_1a_2a_3e^{-(\lambda_2+\lambda_3)t}\\
&+ l_7{a_2}^2a_3e^{-(2\lambda_2+\lambda_3-\lambda_1)t} + l_8{a_1}^2a_4e^{-(\lambda_1+\lambda_3)t}\\
&+ l_9a_1a_2a_4e^{-(\lambda_2+\lambda_3)t} + l_{10}{a_2}^2a_4e^{-(2\lambda_2+\lambda_3-\lambda_1)t}\}
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
\dot{a_2} &= 0\\
\dot{a_3} &= 0
\end{aligned}
$$

$$
\dot{a_4} = \varepsilon\{r_1a_1^2a_4e^{-2\lambda_1 t} + r_2a_1a_2a_4e^{-(\lambda_1+\lambda_2)t} + r_3a_4a_2^2e^{-2\lambda_2 t}\}.
$$

Since a_1, a_2, a_3, a_4 are proportional to the small parameter ε, they are slowly varying functions of time t. Hence they change very little during a period. i.e., they are almost constant in a period. Consequently, we can solve equation (3.14) by assuming a_1, a_2, a_3 and a_4 are constants in the right hand sides of (3.14). Thus the solutions of equations (3.14) are

$$
\begin{aligned}
a_1 = a_{1,0} + \varepsilon\Big\{ & l_1 a_{1,0}{}^3(1-e^{-2\lambda_1 t})/2\lambda_1 \\
& + l_2 a_1{}^2 a_2 (1-e^{-(\lambda_1+\lambda_2)t})/(\lambda_1+\lambda_2) \\
& + l_3 a_1 a_2^2 (1-e^{2\lambda_2 t})/2\lambda_2 + l_4 a_2{}^3(1-e^{-(3\lambda_2-\lambda_1)t})/(3\lambda_2-\lambda_1) \\
& + l_5 a_1{}^2 a_3 (1-e^{-(\lambda_1+\lambda_3)t})/(\lambda_1+\lambda_3) \\
& + l_6 a_1 a_2 a_3 (1-e^{-(\lambda_2+\lambda_3)t})/(\lambda_2+\lambda_3) \\
& + l_7 a_2{}^2 a_3 (1-e^{-(2\lambda_2+\lambda_3-\lambda_1)t})/(2\lambda_2+\lambda_3-\lambda_1) \\
& + l_8 a_1{}^2 a_4 (1-e^{-(\lambda_1+\lambda_3)t})/(\lambda_1+\lambda_3) \\
& + l_9 a_1 a_2 a_4 (1-e^{-(\lambda_2+\lambda_3)t})/(\lambda_2+\lambda_3) \\
& + l_{10} a_2{}^2 a_4 (1-e^{-2(\lambda_2+\lambda_3-\lambda_1)t})/(2\lambda_2+\lambda_3-\lambda_1)\Big\}
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
a_2 &= a_{2,0}, \\
a_3 &= a_{3,0},
\end{aligned}
$$

$$
\begin{aligned}
a_4 = a_{4,0} + \varepsilon\Big\{ & (1-r-1a_1{}^2 a_4 e^{-2\lambda_1 t})/(-2\lambda_1) \\
& + r_2 a_1 a_2 a_4 (1-e^{-(\lambda_1+\lambda_2)t})/(\lambda_1+\lambda_2) \\
& + r_3 a_4 a_2{}^2 (1-e^{-2\lambda_2 t})/2\lambda_2\Big\}.
\end{aligned}
\tag{3.16}
$$

Finally we obtain the first approximation solution of the equation (3.1) as

$$
x(t,\varepsilon) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + (a_3 + a_4 t)e^{-\lambda_3 t} + \varepsilon u_1 \tag{3.17}
$$

where a_1, a_2, a_3, a_4 are given by the equation (3.15) and u_1 is given by the equation (3.7).

4. Results and discussion

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we sometimes compare the approximate solution to the numerical solution. With regard to such a comparison concerning the presented asymptotic solution obtained by KBM method of this paper, we refer the works of Murty et al [5].

For the conditions $\lambda_1 \approx 2\lambda_2, \lambda_1 \approx \lambda_2 + 2\lambda_3$, and $\lambda_3 = \lambda_4$, we have chosen $\lambda_1 = 2.50, \lambda_2 = 1.25, \lambda_3 = \lambda_4 = 1.0$ and $\varepsilon = 0.1$. We have computed $x(t)$ by equation (3.16) in which a_1, a_2, a_3, a_4 are evaluated by the equation (3.15) and u_1 is evaluated by the equation (3.7) with initial conditions $a_1 = 1.0, a_2 = 0.25, a_3 = -0.25, a_4 = 0.0$[or $x(0) = 0.999594, \dot{x}(0) = -2.561182, \ddot{x}(0) = 6.384838, \dddot{x}(0) = -15.828870$] and the results for various time t, have been given in second column of **Table**. Corresponding numerical solutions computed by fourth order Runge-Kutta method, denoted by x_{nu} and have been given in third column of **Table**. The percentage errors have been calculated and given in fourth column. From **Table**, we see that the percentage errors are not greater than 1%. Thus the new asymptotic solution (3.16) agrees with numerical solution nicely.

t	x	x_{nu}	E %
0.00	0.999594	0.999594	0.00000
0.50	0.266584	0.266586	0.00074
1.00	0.061712	0.061713	0.00162
1.50	0.006065	0.006066	0.01648
2.00	-0.006577	-0.006575	-0.03042
2.50	-0.007607	-0.007605	-0.02629
3.00	-0.006014	-0.006013	-0.01663
3.50	-0.004244	-0.004242	-0.04715
4.00	-0.002849	-0.002848	-0.03511
4.50	-0.001863	-0.001863	-0.05371
5.00	-0.001198	-0.001197	-0.08354

5. Conclusion

Krylov -Bogoliubov-Mitropolskii method has been extended to solve fourth order critically damped nonlinear systems under some special conditions. The solutions obtained by this method are very near to the numerical solutions. i.e. the solutions obtained by this method show good agreement with those obtained by numerical method.

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