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ON THE CONVOLUTION OF EXPONENTIAL DISTRIBUTIONS

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Abstract. The distribution of the sum of n independent random variables having exponential distributions with different parameters β_i $(i = 1, 2, ..., n)$ is given in [2], [3], [4] and [6]. In [1], by using Laplace transform, Jasiulewicz and Kordecki generalized the results obtained by Sen and Balakrishnan in [6] and established a formula for the distribution of this sum without conditions on the parameters β_i. The aim of this note is to present a method to find the distribution of the sum of n independent exponentially distributed random variables with different parameters. Our method can also be used to handle the case when all β_i are the same.

1. Introduction

Let $X_1, ..., X_n$ be n independent random variables having exponential distributions with parameters β_i $(i = 1, 2, ..., n)$, i.e., every X_i has a probability density function f_{X_i} given by

$$f_{X_i}(t) := \beta_i \exp(-t\beta_i)\, I_{(0,\infty)}(t), \tag{1.1}$$

for all real number t, where the parameter β_i is positive for all $i = 1, 2, ..., n$ and $I_{(0,\infty)}(t) := 1$ if $t > 0$ and $I_{(0,\infty)}(t) := 0$ for all $t \leq 0$. The sum of these random variables will be denoted by

$$S_n := X_1 + X_2 + ... + X_n. \tag{1.2}$$

The distribution of the random variable S_n is given in [2], [3], [4] and [6]. In the paper [1], H. Jasiulewicz and W. Kordecki generalized the reults obtained by Balakrishnan and Sen in [6]. They computed the distribution of S_n without assuming that all the parameters β_i are different. They reduced the problem to the one of finding the distribution

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of the sum of independent random variables having Erlang distributions. All of the above problems are the special cases of a sum of independent random variables with Gamma distributions. In the paper [5], A.M. Mathai has provided a formula for such a sum. We point out that the method used by the authors in [1] to get their results is based on the use of Laplace transform.

The aim of this note is to present a method by which we compute the convolution of exponential distributions with different parameters. The method is exposed in Section 2. We show in Section 3 that this method can also be used to handle the case where the parameters are the same. Finally, for the sake of completeness, we end this note by recalling (in Section 4) the result of H. Jasiulewicz and W. Kordecki [1] which solves the general case.

2. Convolution of exponential distributions with different parameters

The aim of this section is to present an alternative proof to the following result (see [1], [2], [3], [4], [5] and [6]).

THEOREM 2.1. *Let $X_1, ..., X_n$ be n independent random variables such that every X_i has a probability density function f_{X_i} given by*

$$f_{X_i}(t) := \beta_i \exp(-t\beta_i)\, I_{(0,\infty)}(t) \tag{2.2}$$

for all real number t, where the parameter β_i is positive for all $i = 1, 2, ..., n$. We suppose that the parameters β_i are all distinct. Then the sum S_n has the following probability density function:

$$f_{S_n}(t) = \sum_{i=1}^{n} \frac{\beta_1 \dots \beta_n}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (\beta_j - \beta_i)} \exp(-t\beta_i)\, I_{(0,\infty)}(t), \tag{2.3}$$

for all $t \in \mathbb{R}$.

Our proof will be done by mathematical induction and will use the following algebraic observation.

2.1. Observation.

Let n be a positive integer, let $z_1, \ldots, z_n$ be n different complex numbers, and consider the following rational function:

$$F(z) := \frac{1}{\prod_{j=1}^{n}(z_j - z)}.$$

Then the decomposition of F as a sum of partial fractions gives the following identity:

$$\frac{1}{\prod_{j=1}^{n}(z_j - z)} = \sum_{i=1}^{n} \frac{1}{(z_i - z)\prod_{\substack{j=1\\ j\neq i}}^{n}(z_j - z_i)}. \tag{2.4}$$

2.2. Proof of Theorem 2.1

(a) We start by checking (2.3) for $n = 2$. To this end, let g be any continuous and bounded function on th real line $\mathbb{R}$. Let us compute the following integral:

$$I_2(g) := \int_0^\infty \int_0^\infty g(x_1 + x_2)\beta_1\beta_2 e^{-(\beta_1 x_1 + \beta_2 x_2)} dx_1\, dx_2. \tag{2.5}$$

We set $x_i := {y_i}^2$ for $i = 1, 2$. Then from (2.5) we get

$$I_2(g) = 4\beta_1\beta_2 \int_0^\infty \int_0^\infty g({y_1}^2 + {y_2}^2)\, e^{-(\beta_1 {y_1}^2 + \beta_2 {y_2}^2)} y_1 y_1\, dy_1\, dy_2. \tag{2.6}$$

We compute (2.6) by using the polar coordinates. Thus, we set $y_1 := r\sin\phi$ and $y_2 := r\cos\phi$, where $0 \leq r < \infty$ and $0 \leq \phi \leq \frac{\pi}{2}$. Then from (2.6), by using Fubini's theorem, we get

$$I_2(g) = 4\beta_1\beta_2 \int_0^\infty g(r^2) r^3 \left[\int_0^{\frac{\pi}{2}} e^{-r^2\left(\beta_2 \cos^2\phi + \beta_1 \sin^2\phi\right)} \sin\phi\, \cos\phi\, d\phi\right] dr. \tag{2.7}$$

For every positive number r, we set

$$L_2(r) := \int_0^{\frac{\pi}{2}} e^{-r^2\left(\beta_2 \cos^2\phi + \beta_1 \sin^2\phi\right)} \sin\phi\, \cos\phi\, d\phi. \tag{2.8}$$

Then we have

$$L_2(r) = e^{-r^2\beta_2} \int_0^{\frac{\pi}{2}} e^{-r^2(\beta_1 - \beta_2)\sin^2\phi} \sin\phi\, \cos\phi\, d\phi. \tag{2.9}$$

We observe that

$$\int_0^{\frac{\pi}{2}} e^{-r^2(\beta_1-\beta_2)\sin^2\phi} \sin\phi \, \cos\phi \, d\phi = \frac{1-e^{-r^2(\beta_1-\beta_2)}}{2r^2(\beta_1-\beta_2)}. \tag{2.10}$$

Hence

$$L_2(r) = \frac{e^{-r^2\beta_2} - e^{-r^2\beta_1}}{2r^2(\beta_1-\beta_2)}. \tag{2.11}$$

From (2.7) and (2.11), we obtain

$$I_2(g) = \frac{2\beta_1\beta_2}{\beta_1-\beta_2}\int_0^\infty g(r^2)\left[e^{-r^2\beta_2} - e^{-r^2\beta_1}\right] r\,dr. \tag{2.12}$$

We set $t := r^2$ in (2.12). Then we get

$$I_2(g) = \int_0^\infty g(t)\left[\frac{\beta_1\beta_2}{\beta_2-\beta_1}\left(e^{-t\beta_1} - e^{-t\beta_2}\right)\right] dt. \tag{2.13}$$

The equality (2.13) holding for every continuous and bounded function g on $\mathbb{R}$ shows that the random variable S_2 has a probability density function f_{S_2} given for all real number t by

$$f_{S_2}(t) = \frac{\beta_1\beta_2}{\beta_2-\beta_1}\left(e^{-t\beta_1} - e^{-t\beta_2}\right) I_{(0,\infty)}(t). \tag{2.14}$$

We conclude that the formula (2.3) is satisfied for $n = 2$.

(b) Now we suppose that the formula (2.3) holds true for n and let us prove it for $n+1$. So let $X_1, ..., X_n, X_{n+1}$ be $n+1$ random variables such that every X_i has a density f_{X_i} given by (2.2), where $\beta_1, \dots, \beta_{n+1}$ are distinct. We have $S_{n+1} = S_n + X_{n+1}$. By assumption, we know that S_n and X_{n+1} are independent. Then by a well known result from Probability theory, we deduce that S_{n+1} has a probability density function given for all $t \in \mathbb{R}$ by the convolution

$$f_{S_{n+1}}(t) = f_{S_n} * f_{X_{n+1}}(t). \tag{2.15}$$

then we have

$$f_{S_{n+1}}(t) = \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{\beta_j}{(\beta_j-\beta_i)} f_{X_i} * f_{X_{n+1}}(t), \tag{2.16}$$

for all $t \in \mathbb{R}$. By the part (a) of this proof, we know that

$$f_{X_i} * f_{X_{n+1}}(t) = \frac{\beta_i\beta_{n+1}}{\beta_{n+1}-\beta_i}\left(e^{-t\beta_i} - e^{-t\beta_{n+1}}\right) I_{(0,\infty)}(t), \tag{2.17}$$

for all $t \in \mathbb{R}$. From (2.16) and (2.17), we get the following equalities which hold for all real number t.

$$f_{S_{n+1}}(t) = \sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{\beta_j}{(\beta_j - \beta_i)} \left[\frac{\beta_i \beta_{n+1}}{\beta_{n+1} - \beta_i} \left(e^{-t\beta_i} - e^{-t\beta_{n+1}} \right) \right] I_{(0,\infty)}(t)$$

$$= \sum_{i=1}^{n} \frac{\beta_1 \beta_2 \ldots \beta_{n+1}}{\prod_{\substack{j=1\\ j\neq i}}^{n+1} (\beta_j - \beta_i)} e^{-t\beta_i} I_{(0,\infty)}(t)$$

$$+ \left[\sum_{i=1}^{n} \frac{\beta_1 \beta_2 \ldots \beta_{n+1}}{(\beta_i - \beta_{n+1}) \prod_{\substack{j=1\\ j\neq i}}^{n+1} (\beta_j - \beta_i)} \right] e^{-t\beta_{n+1}} I_{(0,\infty)}(t) \quad (2.18)$$

From the identity (2.4), we know that

$$\frac{1}{\prod_{j=1}^{n} (\beta_j - \beta_{n+1})} = \sum_{i=1}^{n} \frac{1}{(\beta_i - \beta_{n+1}) \prod_{\substack{j=1\\ j\neq i}}^{n} (\beta_j - \beta_i)} \quad (2.19)$$

From (2.18) and (2.19) we deduce that

$$f_{S_{n+1}}(t) = \sum_{i=1}^{n+1} \frac{\beta_1 \beta_2 \ldots \beta_{n+1}}{\prod_{\substack{j=1\\ j\neq i}}^{n+1} (\beta_j - \beta_i)} e^{-t\beta_i} I_{(0,\infty)}(t),$$

for all $t \in \mathbb{R}$. The last equality shows that Theorem 2.1 is completely proved.

2.3. Observations

1. The case $n = 2$ of Theorem 2.1 can be derived directly from the convolution formula. Indeed, $f_{S_2}(y) = 0$ if $y \leq 0$ and for all $y > 0$ we have

$$\begin{aligned} f_{S_2}(y) &= \beta_1\beta_2 \int_0^y e^{-\beta_1(y-x)} e^{-x\beta_2}\, dx \\ &= \beta_1\beta_2\, e^{-y\beta_1} \int_0^y e^{-x(\beta_2-\beta_1)}\, dx \\ &= \frac{\beta_1\beta_2}{\beta_1-\beta_2}\left(e^{-y\beta_2} - e^{-y\beta_1}\right) \end{aligned}$$

which is exactly the formula (2.14).

2. If we fix β_1 and let $\beta_2 \to \beta_1$ in the formula (2.14) then the result for f_{S_2} is the usual Gamma density function for the sum of two independent and identically distributed exponential random variables. This fact can be proved as follows. Consider the mapping $I_2(g)$ defined for all $(\beta_1, \beta_2) \in (0,\infty)^2$ by the formula (2.5). Then $I_2(g)$ is continuous on $(0,\infty)^2$. So, from (2.5) we obtain

$$I_2(g)(\beta_1,\beta_1) = \beta_1^2 \int_0^\infty \int_0^\infty g(x_1+x_2) e^{-\beta_1(x_1+x_2)} dx_1\, dx_2. \qquad (2.20)$$

Now, by using Lebesgue's dominated convergence theorem, we obtain by letting $\beta_2 \to \beta_1$ in (2.13) the following

$$I_2(g)(\beta_1,\beta_1) = \beta_1^2 \int_0^\infty g(t) t e^{-t\beta_1}\, dt. \qquad (2.21)$$

It follows from (2.20) and (2.21) that S_2 has the density given by

$$f_{S_2}(t) = \beta_1^2\, t\, e^{-t\beta_1} I_{(0,\infty)}(t), \qquad (2.22)$$

for all real number t.

Remark. The formula (2.22) is a particular case of the formula (3.1) below.

3. Convolution of exponential distributions with the same parameter

In this section we discuss the case where all β_i are equal to the same positive number β. By using the moment generating functions, it can be shown that the distribution of the sum S_n is a Gamma distribution. Next, we show how our method can be used to handle this case.

THEOREM 3.1. *Suppose that all the β_i are equal to a number $\beta > 0$, then the random variable S_n has a probability density function f_{S_n} given by*

$$f_{S_n}(t) = \frac{\beta^n \, t^{n-1}}{(n-1)!} \, e^{-t\beta} I_{(0,\infty)}(t), \tag{3.1}$$

for all real number t.

Proof. As in Theorem 1, we are led to compute the integrals

$$I_n(g) := \beta^n \int_0^\infty ... \int_0^\infty g(x_1 + ... + x_n) e^{-\beta(x_1+...+x_n)} dx_1 ... dx_n \tag{3.2}$$

for all $g \in \mathcal{C}_b(\mathbb{R})$ the space of all continuous and bounded functions on the real line. To compute (3.2), we set $x_i := {y_i}^2$ where $0 \leq y_i < \infty$ for all $i = 1, 2, ..., n$. With these new variables we have

$$I_n(g) = 2^n \beta^n \int_0^\infty ... \int_0^\infty g({y_1}^2 + ... + {y_n}^2) \, y_1 \dots y_n e^{-\beta({y_1}^2+...+{y_n}^2)} dy_1 ... dy_n. \tag{3.3}$$

To compute the integral (3.3), we shall use the spherical coordinates. Thus, we set

$$\begin{aligned} y_1 &= r \sin\phi_{n-1} ... \sin\phi_3 \sin\phi_2 \sin\phi_1 \\ y_2 &= r \sin\phi_{n-1} ... \sin\phi_3 \sin\phi_2 \cos\phi_1 \\ y_3 &= r \sin\phi_{n-1} ... \sin\phi_3 \cos\phi_2 \\ ... &= .... \\ y_{n-1} &= r \sin\phi_{n-1} \cos\phi_{n-2} \\ y_n &= r \cos\phi_{n-1}, \end{aligned}$$

where $0 \leq r < \infty$, and $0 \leq \phi_k \leq \frac{\pi}{2}$ for all $k = 1, ...n-1$. Conversely, for all $k = 1, 2, ..., n-1$, by setting $r_k^2 := y_1^2 + ... + y_k^2$ and $r_n := r$, we have

$$\cos(\phi_k) = \frac{y_{k+1}}{r_{k+1}},$$

and

$$\sin(\phi_k) = \frac{r_k}{r_{k+1}}.$$

We recall that the Jacobian of this change of variables is given by

$$r^{n-1} \sin^{n-2}\phi_{n-1} \sin^{n-3}\phi_{n-2} ... \sin\phi_2. \tag{3.4}$$

With these spherical coordinates, we have

$$\prod_{i=1}^{n} y_i = r^n \prod_{j=1}^{n-1} \sin^j(\phi_j)\cos(\phi_j). \tag{3.5}$$

By using Fubini's theorem and the previous identities, we have

$$I_n(g) = 2^n\beta^n \left[\prod_{k=1}^{n-1} \int_0^{\frac{\pi}{2}} \sin^{2k-1}(\phi_k)\cos(\phi_k)\, d\phi_k\right] \int_0^{\infty} g(r^2) r^{2n-1} e^{-\beta r^2}\, dr$$

$$= \frac{2\beta^n}{(n-1)!} \int_0^{\infty} g(r^2) r^{2n-1} e^{-\beta r^2}\, dr \tag{3.6}$$

By setting $t = r^2$ in (3.6), we have

$$\int_0^{\infty} g(r^2) r^{2n-1} e^{-\beta r^2}\, dr = \frac{1}{2} \int_0^{\infty} g(t) t^{n-1} e^{-t\beta}\, dt. \tag{3.7}$$

Therefore

$$I_n(g) = \frac{\beta^n}{(n-1)!} \int_0^{\infty} g(t) t^{n-1} e^{-t\beta}\, dt. \tag{3.8}$$

(3.8) shows that the random variable S_n has a probability density function f_{S_n} given by

$$f_{S_n}(t) = \frac{\beta^n t^{n-1}}{(n-1)!} e^{-t\beta} I_{(0,\infty)}(t).$$

Thus our theorem is completely proved. □

4. Convolution of exponential distributions: General case

Let Y_i be independent random variables having exponential distributions with parameters β_i, for $i = 1, 2, \ldots r$. The parameters do not have to be different. We want to find the distribution of the random variable

$$S_r := Y_1 + Y_2 + \ldots + Y_r. \tag{4.1}$$

Suppose that there are n different parameters from among $\beta_1, \beta_2, \ldots, \beta_r$ where $1 < n < r$. Without loss of generality, one can assume that these different parameters are $\beta_1, \beta_2, \ldots, \beta_n$. The components in the sum S_r are grouped with respect to the parameter β_i for $i = 1, 2, \ldots, n$. Let k_i denote the number of the components having the same parameter β_i. We have $r = k_1 + k_2 + \ldots + k_n$. We denote X_i the sum of the components having the same parameter β_i. It is known (see also Section 3) that the random variable X_n has an Erlang distribution $Erl(k_i, \beta_i)$, i.e., its density is that one given by the formula (3.1) with $n = k_i$ and $\beta = \beta_i$.

Then one can write S_r as the sum of n independent random variables having $Erl(k_i, \beta_i)$ distributions $(i = 1, 2, \ldots, n)$ i.e.

$$S_r := X_1 + X_2 + \ldots + X_n, \tag{4.2}$$

where $\beta_i \neq \beta_j$ for $i \neq j$.

The distribution of S_r is given by the following result established by H. Jasiulewicz and W. Kordecki in [1].

THEOREM 4.1. *The probability density function f_{S_r} of the random variable S_r is given by*

$$f_{S_r}(t) = \sum_{i=1}^{n} \beta_i^{k_i} e^{-t\beta_i} \sum_{j=1}^{k_i} \frac{(-1)^{k_i-j}}{(j-1)!} t^{j-1}$$
$$\times \sum_{\substack{m_1+\ldots+m_n=k_i-j \\ m_i=0}} \prod_{\substack{l=1 \\ l\neq i}}^{n} \binom{k_l+m_l-1}{m_l} \frac{{\beta_l}^{k_l}}{(\beta_l-\beta_i)^{k_l+m_l}}, \tag{4.3}$$

for $t > 0$ and $f_{S_r}(t) = 0$ for $t \leq 0$.

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References

[1] H. Jasiulewicz and W. Kordecki, *Convolutions of Erlang and of Pascal distributions with applications to reliability,* Demonstratio Math. **36** (2003), no. 1, 231-238

[2] N. L. Johnson and S. Kotz and N. Balakrishnan, *Continuous Univariate Distributions -1,* Wiley, New York, 1994.

[3] W. Kordecki, *Reliability bounds for multistage structure with independent compnents,* Stat. Probab. Lett. **34**, 43-51.

[4] M. V. Lomonosov, *Bernoulli scheme with closure,* Problems Inform. Transmission, **10**, (1974) 73-81.

[5] A. M. Mathai, *Storage capacity of a dam with gamma type inputs,* Ann. Inst. Statist. Math. **34** (1982), 591-597.

[6] A. Sen and N. Balakrishnan, *Convolutions of geometrics and a reliability problem,* Stat. Probab. Lett. **43** (1999), 421-426.

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