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SUBMANIFOLDS OF AN ALMOST r-PARACONTACT RIEMANNIAN MANIFOLD ENDOWED WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. We define a semi-symmetric non-metric connection in an almost r-paracontact Riemannian manifold and we consider submanifolds of an almost r-paracontact Riemannian manifold endowed with a semi-symmetric non-metric connection and obtain Gauss and Codazzi equations, Weingarten equation and curvature tensor for submanifolds of an almost r-paracontact Riemannian manifold endowed with a semi-symmetric non-metric connection.

1. Introduction

In [10], R. S. Mishra studied almost complex and almost contact submanifolds. In [11], R. Nivas considered submanifols of a Riemannian manifold with semi-symmetric connection. Some properties of submanifolds of a Riemannian manifold with semi-symmetric semi-metric connection were studied in [4] by B. Barua. Moreover, In [9], I. Mihai and K. Motsumoto studied submanifolds of an almost r-paracontact Riemannian manifold of P-Sasakian type.

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

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The connection ∇ is *symmetric* if its torsion tensor T vanishes, otherwise it is non-symmeteic. The connection ∇ is *metric* if there is a Riemannian metric g in M is such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

In [6], A. Friedmann and J. A. Schouten introduced the idea of a semi-symmetric linear connection in a differentiable manifold. A linear connection is said to be *semi-symmetric* if its torsion tensor T is of the form

$$T(X,Y) = u(Y)X - u(X)Y,$$

where u is a 1-form. In [1], [2], [3], [7], [8] and [12], some kinds of semi-symmetric connections were studied.

Let M be an n-dimensional Riemannian manifold with a positive definite metric g. If there exist a tensor field ϕ of type (1,1), r vector fields $\xi_1, \xi_2, ..., \xi_r$ $(n > r)$, r 1-forms $\eta^1, \eta^2, ..., \eta^r$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad \alpha, \beta \in (r) = \{1, 2, \ldots r\} \tag{i}$$

$$\phi^2(X) = X - \eta^\alpha(X)\xi_\alpha \tag{ii}$$

$$\eta^\alpha(X) = g(X, \xi_\alpha), \quad \alpha \in (r) \tag{iii}$$

$$g(\phi X, \phi Y) = g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y), \tag{iv}$$

where X and Y are vector fields on M, then the structure $(\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be an *almost r-paracontact Riemannian structure* and M is an *almost r-paracontact Riemannian manifold* [2].
From (i) through (iv), we have

$$\phi(\xi_\alpha) = 0, \qquad \alpha \in (r) \tag{v}$$

$$\eta^\alpha \circ \phi = 0, \qquad \alpha \in (r) \tag{vi}$$

$$\Psi(X,Y) \stackrel{\text{def}}{=} g(\phi X, Y) = g(X, \phi Y). \tag{vii}$$

An almost r-paracontact Riemannian manifold M with structure $(\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be *S-paracontact manifold* if

$$\Psi(X,Y) = (\dot{\nabla}_Y \eta^\alpha)(X), \qquad \text{for all} \quad \alpha \in (r).$$

An almost r-paracontact Riemannian manifold M with structure $(\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be *P-Sasakian manifold* if it also satisfies

$$\dot{\nabla}_Z \Psi(X,Y) = -\sum_\alpha \eta^\alpha(X)[g(Y,Z) - \sum_\beta \eta^\beta(Y)\eta^\beta(Z)]$$

$$-\sum_{\alpha}\eta^{\alpha}(Y)[g(X,Z)-\sum_{\beta}\eta^{\beta}(X)\eta^{\beta}(Z)]$$

for all vector fields X, Y and Z on M [9].
The above two conditions are respectively equivalent to

$$\phi X = \dot{\nabla}_X \xi_\alpha, \qquad \text{for all} \quad \alpha \in (r) \quad \text{and}$$

$$\dot{\nabla}_Y\phi(X) = -\sum_{\alpha}\eta^{\alpha}(X)[Y-\eta^{\alpha}(Y)\xi_\alpha]-[g(X,Y)-\sum_{\alpha}\eta^{\alpha}(X)\eta^{\alpha}(Y)]\sum_{\beta}\xi_\beta.$$

In this paper, we study semi-symmetric non-metric connection in an almost r-paracontact Riemannian manifold. We consider hypesurfaces and submanifolds of almost r-paracontact Riemannian manifold endowed with a semi-symmetric non-metric connection. We also obtain Gauss and Codazzi equations for hypersurfaces and curvature tensor and Weingarten equation for submanifolds of almost r-paracontact Riemannian manifold with respect to semi-symmetric non-metric connection.

2. Preliminaries

Let M^{n+1} be an $(n+1)$-dimensional differentiable almost r-paracontact Riemannian manifold of class C^∞ and M^n be the hypersurface in M^{n+1} by the immersion τ: $M^n \to M^{n+1}$. The differential $d\tau$ of the immersion τ is denoted by B. The vector field X in the tangent space of M^n corresponds to a vector field BX in that of M^{n+1}. Suppose that $\tilde{g}$ be the metric in the enveloping manifold M^{n+1} and g the induced metric of hypersurface M^n defined by

$$g(\phi X, Y) = \tilde{g}(B\phi X, BY),$$

where X and Y are the arbitrary vector fields and ϕ is a tensor of type (1,1). If the Riemannian manifolds M^{n+1} and M^n are both orientable, we can choose a unique vector field N defined along M^n such that $\tilde{g}(BX,N)=0$ and $\tilde{g}(N,N)=1$ for arbitrary vector field N in M^n. We call this vector field the normal vector field to the hypersurface M^n.

We now define a semi-symmetric non-metric connection $\tilde{\nabla}$ by [2]

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \tilde{\dot{\nabla}}_{\tilde{X}}\tilde{Y} + \tilde{\eta}^{\alpha}(\tilde{Y})\tilde{X} \tag{2.1}$$

for arbitrary vector fields $\tilde{X}$ and $\tilde{Y}$ tangents to M^{n+1}, where $\tilde{\nabla}$ denotes the Levi-Civita connection with respect to Riemannian metric $\tilde{g}$, $\tilde{\eta}^\alpha$ is a 1-form, and $\tilde{\xi_\alpha}$ is the vector field defined by

$$\tilde{g}(\tilde{\xi_\alpha}, \tilde{X}) = \tilde{\eta}^{\alpha}(\tilde{X})$$

for arbitrary vector fields $\tilde{X}$ on M^{n+1}. Also

$$\tilde{g}(\tilde{\phi}\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{\phi}\tilde{Y}),$$

where $\tilde{\phi}$ is a (1,1)-tensor field.

Now, suppose that $(\tilde{\phi}, \tilde{\xi}_\alpha, \tilde{\eta}^\alpha, \tilde{g})_{\alpha\in(r)}$ is an almost r-paracontact Riemannian structure on M^{n+1}. Then every vector field $\tilde{X}$ on M^{n+1} is decomposed as

$$\tilde{X} = BX + \ell(\tilde{X})N,$$

where ℓ is a 1-form on M^{n+1} and for any vector field X on M^n and normal N. Also we have $b(BX) = b(X)$, $\phi(BX) = B\phi(X)$, where b is a 1-form on M^n.

For each $\alpha \in (r)$, we have [2]

$$\tilde{\phi}BX = B\phi X + b(X)N, \tag{2.2}$$

$$\tilde{\xi}_\alpha = B\xi_\alpha + a_\alpha N, \tag{2.3}$$

where ξ_α is a vector field and a_α is defined as

$$a_\alpha = m(\xi_\alpha) = \eta^\alpha(N) \tag{2.4}$$

for each $\alpha \in (r)$ on M^n. Now we defined $\tilde{\eta}^\alpha$ as

$$\tilde{\eta}^\alpha(BX) = \eta^\alpha(X). \tag{2.5}$$

Then we can know the following.

THEOREM 2.1. *The connection induced on the hypersurface M^n of an almost r-paracontact Riemannian manifold M^{n+1} with a semi-symmetric non-metric connection with respect to the unit normal is also a semi-symmetric non-metric connection.*

Proof. Let $\dot{\nabla}$ be the induced connection from $\dot{\tilde{\nabla}}$ on the hypersurface M^n with respect to the unit normal N. Then we have

$$\dot{\tilde{\nabla}}_{BX}BY = B\dot{\nabla}_X Y + h(X,Y)N \tag{2.6}$$

for arbitrary vector fields X and Y of M^n, where h is a second fundamental tensor of the hypersurface M^n. Let ∇ be connection induced on the hypersurface from $\tilde{\nabla}$ with respect to the normal N. Then we have

$$\tilde{\nabla}_{BX}BY = B\nabla_X Y + m(X,Y)N \tag{2.7}$$

for arbitrary vector fields X and Y of M^n, where m being a tensor field of type (0,2) on the hypersurface of M^n.
From equation (2.1), we have

$$\tilde{\nabla}_{BX}BY = \dot{\tilde{\nabla}}_{BX}BY + \tilde{\eta}^{\alpha}(BY)\tilde{\phi}BX.$$

Using (2.5), (2.6) and (2.7) in the above equation, we get

$$B(\nabla_X Y) + m(X,Y)N = B\dot{\nabla}_X Y + h(X,Y)N + \eta^{\alpha}(Y)BX. \tag{2.8}$$

Comparison of tangential and normal vector fields yields,

$$\nabla_X Y = \dot{\nabla}_X Y + \eta^{\alpha}(Y)X \tag{2.9}$$

and

$$m(X,Y) = h(X,Y). \tag{2.10}$$

Thus

$$\nabla_X Y - \nabla_Y X - [X,Y] = \eta^{\alpha}(Y)X - \eta^{\alpha}(X)Y. \tag{2.11}$$

Hence the connection ∇ induced on M^n is a semi-symmetric non-metric connection [7]. □

3. Totally geodesic and totally umbilical hypersurfaces

We define $\dot{\nabla}B$ and ∇B respectively by

$$(\dot{\nabla}B)(X,Y) = (\dot{\nabla}_X B)(Y) = \dot{\tilde{\nabla}}_{BX}BY - B(\dot{\nabla}_X Y)$$

and

$$(\nabla B)(X,Y) = (\nabla_X B)(Y) = (\tilde{\nabla}_{BX}BY) - B(\nabla_X Y),$$

where X and Y being arbitrary vector fields on M^n. Then equations (2.6) and (2.7) take the form

$$(\dot{\nabla}_X B)(Y) = h(X,Y)N$$

and

$$(\nabla_X B)(Y) = m(X,Y)N.$$

These are Gauss equations with respect to induced connection $\dot{\nabla}$ and ∇ respectively.

Let X_1, X_2, ..., X_n be n-orthonormal vector fields, then the function

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i, X_i)$$

is called the mean curvature of M^n with respect to Riemannian connection $\dot{\nabla}$ and

$$\frac{1}{n}\sum_{i=1}^{n} m(X_i, X_i)$$

is called the mean curvature of M^n with respect to the semi-symmetric non-metric connection ∇.
From these we define the followings.

Definition 3.1. *The hypersurface M^n is called totally geodesic hypersurface of M^{n+1} with respect to the Riemannian connection $\dot{\nabla}$ if h vanishes.*

Definition 3.2. *The hypersurface M^n is called totally umbilical with respect to connection $\dot{\nabla}$ if h is proportional to the metric tensor g.*

We call M^n is totally geodesic and totally umbilical with respect to semi-symmetric non-metric connection ∇ according as the function m vanishes and proportional to the metric g respectively.
Then we have following theorems.

Theorem 3.3. *The mean curvature of the hypersurface M^n of an almost r-paracontact Riemannian manifold M^{n+1} with respect to the Riemannian connection $\dot{\nabla}$ coincides with that of M^n with respect to semi-symmetric non-metric connection ∇.*

Proof. In view of (2.10) we have

$$m(X_i, X_i) = h(X_i, X_i).$$

Summing up for $i = 1, 2, ..., n$ and dividing by n, we obtain

$$\frac{1}{n}\sum_{i=1}^{n} m(X_i, X_i) = \frac{1}{n}\sum_{i=1}^{n} h(X_i, X_i),$$

which proves the theorem. □

Theorem 3.4. *The hypersurface M^n of an almost r-paracontact Riemannian manifold M^{n+1} is totally geodesic with respect to the Riemannian connection $\dot{\nabla}$ if and only if it is totally geodesic with respect to the semi-symmetric non-metric connection.*

Proof. The proof follows from (2.10) easily. □

4. Gauss, Weingarten and Codazzi equations

In this section we shall obtain Weingarten equation with respect to the semi-symmetric non-metric connection ∇. For the Riemannian connection $\dot{\nabla}$, these equations are given by

$$\dot{\tilde{\nabla}}_{BX}N = -BHX \tag{4.1}$$

for any vector field X in M^n, where h is a tensor field of type (1,1) of M^n defined by

$$g(HX,Y) = h(X,Y). \tag{4.2}$$

From equations (2.1), (2.2) and (2.4) we have

$$\tilde{\nabla}_{B\tilde{X}}N = \dot{\tilde{\nabla}}_{B\tilde{X}}N + a_\alpha BX. \tag{4.3}$$

Using (4.1) we have

$$\tilde{\nabla}_{B\tilde{X}}N = -BMX, \tag{4.4}$$

where $MX = (H - a_\alpha)X$ for any vector field X in M^n. Equation (4.4) is the Weingarten equation.

We shall find equation of Gauss and Codazzi with respect the semi-symmetric non-metric connection. The curvature tensor with respect to semi-symmetric non-metric connection $\tilde{\nabla}$ of M^{n+1} is

$$\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z}. \tag{4.5}$$

Putting $\tilde{X} = BX$, $\tilde{Y} = BY$ and $\tilde{Z} = BZ$, we have

$$\tilde{R}(BX,BY)BZ = \tilde{\nabla}_{BX}\tilde{\nabla}_{BY}BZ - \tilde{\nabla}_{BY}\tilde{\nabla}_{BX}BZ - \tilde{\nabla}_{[BX,BY]}BZ.$$

By virtue of (2.7), (2.11) and (4.4), we get

$$\begin{aligned}\tilde{R}(BX,BY)BZ = B\{R(X,Y)Z + m(X,Z)MY - m(Y,Z)MX\} \\ +\{(\nabla_X m)(Y,Z) - (\nabla_Y m)(X,Z)\} \\ +\{m(\eta^\alpha(Y)X - \eta^\alpha(X)Y), Z\}N,\end{aligned} \tag{4.6}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

is the curvature tensor of the semi-symmetric non-metric connection ∇. We denote

$$\tilde{R}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{U}) = g(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{U})$$

and

$$R(X,Y,Z,U) = g(R(X,Y)Z,U).$$

Then from (4.6), we can easily show that

$$\tilde{R}(BX,BY,BZ,BU) = R(X,Y,Z,U) + m(X,Z)m(Y,U)$$

$$-m(Y,Z)m(X,U)+a_\alpha g(X,U)m(Y,Z)-a_\alpha g(Y,U)m(X,Z) \quad (4.7)$$

and

$$\tilde{R}(BX,BY,BZ,N)=(\nabla_X m)(Y,Z)-(\nabla_Y m)(X,Z) \quad (4.8)$$
$$+\eta^\alpha(Y)m(X,Z)-\eta^\alpha(X)m(Y,Z).$$

Equation (4.7) and (4.8) are the equation of Gauss and Codazzi with respect to the semi-symmetric non-metric connection respectively.

5. Submanifolds of codimension 2

Let M^{n+1} be an $(n+1)$-dimensional differentiable almost r-paracontact Riemannian manifold of differentiability class C^∞ and M^{n-1} be an $(n-1)$-dimensional submanifold immersed in M^{n+1} by immersion τ: $M^{n-1}\to M^{n+1}$. We denote the differential $d\tau$ of the immersion τ by B, so that the vector field X in the tangent space of M^{n-1} corresponds to a vector field BX in that of M^{n+1}. Suppose that $\tilde{g}$ be the metric in the enveloping manifold M^{n+1} and g the induced metric of submanifold M^{n-1} defined by

$$\tilde{g}(B\phi X,BY)=g(\phi X,Y)$$

for any arbitrary vector fields X and Y in M^{n-1} [11]. Let the manifolds M^{n+1} and M^{n-1} are both orientable such that

$$\tilde{g}(B\phi X,N_1)=\tilde{g}(B\phi X,N_2)=\tilde{g}(N_1,N_2)=0$$

and

$$\tilde{g}(N_1,N_1)=\tilde{g}(N_2,N_2)=1$$

for arbitrary vector field X in M^{n-1} and two unit normals N_1 and N_2 to M^{n-1} [6]. We suppose that the enveloping manifold M^{n+1} admits a semi-symmetric non-metric connection ∇ given by [1]

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y}=\dot{\tilde{\nabla}}_{\tilde{X}}\tilde{Y}+\tilde{\eta}^\alpha(\tilde{Y})\tilde{X}$$

for arbitrary vector fields $\tilde{X}$ and $\tilde{Y}$ in M^{n-1}, $\dot{\tilde{\nabla}}$ denotes the Riemannian connection with respect to the Riemannian metric $\tilde{g}$ and $\tilde{\eta}^\alpha$ is a 1-form. Let us now put

$$\tilde{\phi}BX=B\phi X+a(X)N_1+b(X)N_2 \quad (5.1)$$

$$\tilde{\xi}_\alpha=B\xi_\alpha+a_\alpha N_1+b_\alpha N_2, \quad (5.2)$$

where $a(X)$ and $b(X)$ are 1-forms on M^{n-1}, ξ_α is a vector field in the tangent space on M^{n-1}, and a_α, b_α are functions on M^{n-1} defined by

$$\eta^\alpha(N_1)=a_\alpha,\quad \eta^\alpha(N_2)=b_\alpha. \quad (5.3)$$

Then we can prove the following.

THEOREM 5.1. *The connection induced on the submanifold M^{n-1} of an almost r-paracontact Riemannian manifold M^{n+1} with semi-symmetric non-metric connection ∇ is also a semi-symmetric non-metric connection.*

Proof. Let $\dot{\nabla}$ be the connection induced on the submanifolds M^{n-1} from the connection $\tilde{\dot{\nabla}}$ on the enveloping manifold with respect to unit normals N_1 and N_2, then we have [9]

$$\tilde{\dot{\nabla}}_{BX}BY = B(\dot{\nabla}_X Y) + h(X,Y)N_1 + k(X,Y)N_2 \tag{5.4}$$

for arbitrary vector fields X and Y of M^{n-1}, where h and k are second fundamental tensors of M^{n-1}. Similarly, if ∇ is the connection induced on M^{n-1} from the semi-symmetric non-metric connection $\tilde{\nabla}$ on M^{n-1}, we have

$$\tilde{\nabla}_{BX}BY = B(\nabla_X Y) + m(X,Y)N_1 + n(X,Y)N_2, \tag{5.5}$$

where m and n being tensor fields of type (0,2) of the submanifold M^{n-1}. In view of equation (2.1), we have

$$\tilde{\nabla}_{BX}BY = \tilde{\dot{\nabla}}_{BX}BY + \tilde{\eta}^{\alpha}(BY)(BX).$$

In view of equations (5.1), (5.2), (5.4) and (5.5), we have

$$B(\nabla_X Y)+m(X,Y)N_1+n(X,Y)N_2 = B(\dot{\nabla}_X Y)+h(X,Y)N_1+k(X,Y)N_2 \\ +\eta^{\alpha}(Y)BX, \tag{5.6}$$

where $\tilde{\eta}^{\alpha}(BY) = \tilde{\eta}^{\alpha}(Y)$ and $g(BX,BY) = g(X,Y)$.
Comparing tangential and normal vector fields to M^{n-1}, we get

$$\nabla_X Y = \dot{\nabla}_X Y + \eta^{\alpha}(Y)X, \tag{5.7}$$

$$m(X,Y) = h(X,Y) \tag*{$(5.8)_a$}$$

and

$$n(X,Y) = k(X,Y). \tag*{$(5.8)_b$}$$

Thus

$$\nabla_X Y - \nabla_Y X - [X,Y] = \eta^{\alpha}(Y)X - \eta^{\alpha}(X)Y. \tag{5.9}$$

Hence the connection ∇ induced on M^{n-1} is semi-symmetric non-metric connection. □

6. Totally geodesic and totally umbilical submanifolds

Let X_1, X_2, ..., X_n be $(n-1)$-orthonormal vector fields on the submanifold M^{n-1}. Then the function

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}\{h(X_i,X_i)+k(X_i,X_i)\}$$

is the mean curvature of M^{n-1} with respect to the Riemannian connection $\dot{\nabla}$ and

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}\{m(X_i,X_i)+n(X_i,X_i)\}$$

is the mean curvature of M^{n-1} with respect to ∇ [6].
Now we define the followings.

DEFINITION 6.1. *If h and k vanish separately, the submanifold M^{n-1} is called totally geodesic with respect to the Riemannian connection $\dot{\nabla}$.*

DEFINITION 6.2. *The submanifold M^{n-1} is called totally umbilical with respect to the Riemannian connection $\dot{\nabla}$ if h and k are proportional to the metric g.*

We call M^{n-1} is totally geodesic and totally umbilical with respect to the semi-symmetric non-metric connection ∇ according as the functions m and n vanish separately and are proportional to metric tensor g respectively.
Then we can prove the following.

THEOREM 6.3. *The mean curvature of submanifold M^{n-1} of an almost r-paracontact Riemannian manifold M^{n+1} with respect to the Riemannian connection $\dot{\nabla}$ coincides with that of M^{n-1} with respect to the semi-symmetric non-metric connection ∇.*

Proof. In view of (5.8) we have

$$m(X_i,X_i)+n(X_i,X_i)=h(X_i,X_i)+k(X_i,X_i).$$

Summing up for $i=1,2,...,n-1$ and dividing by $2(n-1)$, we get

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}\{m(X_i,X_i)+n(X_i,X_i)\}=\frac{1}{2(n-1)}\sum_{i=1}^{n-1}\{h(X_i,X_i)+k(X_i,X_i)\},$$

which proves our assertion. □

THEOREM 6.4. *The submanifold M^{n-1} of an almost r-paracontact Riemannian manifold M^{n+1} is totally geodesic with respect to the Riemannian connection $\dot{\nabla}$ if and only if it is totally geodesic with respect to the semi-symmetric non-metric connection ∇.*

Proof. The proof follows easily from equations $(5.8)_a$ and $(5.8)_b$. □

7. Curvature tensor and Weingarten equations

For Riemannian connection $\dot{\nabla}$, the Weingarten equations are given by [9]

$$\dot{\nabla}_{BX} N_1 = -BHX + l(X)N_2 \tag{7.1}$$

and

$$\dot{\nabla}_{BX} N_2 = -BKX - l(X)N_1,$$

where H and K are tensor fields of type (1,1) such that

$$g(HX,Y) = h(X,Y) \tag{7.2}$$

and

$$g(KX,Y) = k(X,Y)$$

and also l is a tensor fields of type $(1,1)$. Furthermore making use of (2.1) and (7.1), we get

$$\tilde{\nabla}_{BX} N_1 = -B(H - a_\alpha)X + l(X)N_2,$$
$$\tilde{\nabla}_{BX} N_1 = -BM_1X + l(X)N_2, \tag{7.3}$$

where $M_1 \equiv H - a_\alpha$. Similarly from (2.1) and (7.1) we can also get

$$\tilde{\nabla}_{BX} N_2 = -BM_2X, \tag{7.4}$$

where $M_2 \equiv K - b_\alpha$. Equations (7.3) and (7.4) are Weingarten equations with respect to the semi-symmetric non-metric connection ∇.

8. Riemannian curvature tensor for semi-symmetric non-metric connection

Let $\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z}$ be the Riemannian curvature tensor of the enveloping manifold M^{n+1} with respect to the semi-symmetric non-metric connection ∇, then

$$\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z}.$$

Replacing $\tilde{X}$ by BX, $\tilde{Y}$ by BY and $\tilde{Z}$ by BZ, we get

$$\tilde{R}(BX,BY)BZ = \tilde{\nabla}_{BX}\tilde{\nabla}_{BY}BZ - \tilde{\nabla}_{BY}\tilde{\nabla}_{BX}BZ - \tilde{\nabla}_{[BX,BY]}BZ.$$

Using (7.3), we get

$$\begin{aligned}\tilde{R}(BX,BY)BZ = & \tilde{\nabla}_{BX}\{B(\nabla_Y Z) + m(Y,Z)N_1 + n(Y,Z)N_2\} \\ & -\tilde{\nabla}_{BY}\{B(\nabla_X Z) + m(X,Z)N_1 + n(X,Z)N_2\} \\ & -\{B(\nabla_{[X,Y]}Z) + m([X,Y],Z)N_1 + n([X,Y],Z)N_2\}.\end{aligned}$$

Again using (5.5), (7.3), (7.4) and (5.9), we have

$$\begin{aligned}\tilde{R}(BX,BY)BZ = & BR(X,Y,Z) + B\{m(X,Z)M_1Y - m(Y,Z)M_1X \\ & +n(X,Z)M_2Y - n(Y,Z)M_2X\} \\ & +m\{\eta^\alpha(Y)X - \eta^\alpha(X)Y, Z\}N_1 \\ & +n\{\eta^\alpha(Y)X - \eta^\alpha(X)Y, Z\}N_2 \\ & +\{(\nabla_X m)(Y,Z) - (\nabla_Y m)(X,Z)\}N_1 \\ & +\{(\nabla_X n)(Y,Z) - (\nabla_Y n)(X,Z)\}N_2 \\ & +l(X)\{m(Y,Z)N_2 - n(Y,Z)N_1\} \\ & -l(Y)\{m(X,Z)N_2 - n(X,Z)N_1\},\end{aligned}$$

where $R(X,Y,Z)$ being the Riemannian curvature tensor of the submanifold with respect to the semi-symmetric non-metric connection ∇.

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