JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **24**, No. 1, March 2011

SUBMANIFOLDS OF AN ALMOST r-PARACONTACT RIEMANNIAN MANIFOLD ENDOWED WITH A QUARTER-SYMMETRIC NON-METRIC CONNECTION

Mobin Ahmad*, Jae-Bok Jun**, and Abdul Haseeb***

Abstract. We define a quarter-symmetric non-metric connection in an almost r-paracontact Riemannian manifold and we consider the submanifolds of an almost r-paracontact Riemannian manifold endowed with a quarter-symmetric non-metric connection. We also obtain the Gauss, Codazzi and Weingarten equations and the curvature tensor for the submanifolds of an almost r-paracontact Riemannian manifold endowed with a quarter-symmetric non-metric connection.

1. Introduction

In [9], R. S. Mishra studied almost complex and almost contact submanifolds. And in [3], S. Ali and R. Nivas considered submanifolds of a Rimannian manifold with a quarter-symmetric connection. Some properties of submanifolds of a Riemannian manifold with a quarter-symmetric semi-metric connection were studied in [6] by L. S. Dass etc. Moreover, in [8], I. Mihai and K. Matsumoto studied the submanifolds of an almost r-paracontact Riemannian manifold of P-Sasakian type.

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Received December 31, 2010; Accepted February 15, 2011.
2010 Mathematics Subject Classification: Primary 53C05, 53D12.
Key words and phrases: hypersurfaces, submanifolds, almost r-paracontact manifold, quarter-symmetric non-metric connection, Gauss, Weingarten and Codazzi equations.
Correspondence should be addressed to Jae-Bok Jun, jbjun@kookmin.ac.kr.

The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is Levi-Civita connection.

In [7], S. Golab introduced the idea of a quarter-symmetric linear connection if its torsion tensor T is of the form

$$T(X, Y) = u(Y)\psi X - u(X)\psi Y,$$

where u is a 1-form and ψ is a tensor field of type (1,1). In [10], R. S. Mishra and S. N. Pandey considered a quarter-symmetric metric connection and studied some of its properties. In [1], [2], [4], [11], [12] and [13], some kinds of quarter-symmetric metric connections were studied.

Let M be an n-dimensional Riemannian manifold with a positive definite metric g. If there exist a tensor field ψ of type (1,1), r-vector fields $\xi_1, \xi_2, ..., \xi_r$ $(n > r)$, r 1-forms $\eta^1, \eta^2, ..., \eta^r$ such that

(i) $\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad \alpha, \beta \in (r) = \{1, 2, ..., r\}$,

(ii) $\psi^2(X) = X - \eta^\alpha(X)\xi_\alpha$,

(iii) $\eta^\alpha(X) = g(X, \xi_\alpha), \quad \alpha \in (r)$,

(iv) $g(\psi X, \psi Y) = g(X, Y) - \Sigma_\alpha \eta^\alpha(X)\eta^\alpha(Y)$,

where X and Y are vector fields on M and $a^\alpha b_\alpha \overset{\text{def}}{=} \Sigma_\alpha a^\alpha b_\alpha$, then the structure $\Sigma = (\psi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ is said to be an *almost r-paracontact Riemannian structure* on M and M is an *almost r-paracontact Riemannian manifold* [1].

With the help of the above conditions (i), (ii), (iii) and (iv) we have

(v) $\psi(\xi_\alpha) = 0, \quad \alpha \in (r)$,

(vi) $\eta^\alpha \circ \psi = 0, \quad \alpha \in (r)$,

(vii) $\Psi(X, Y) \overset{\text{def}}{=} g(\psi X, Y) = g(X, \psi Y)$.

An almost r-paracontact Riemannian manifold M with a structure $\Sigma = (\psi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ is said to be of *S-paracontact type* [1] if

$$\Psi(X, Y) = (\nabla^*_Y \eta^\alpha)(X), \quad \alpha \in (r).$$

An almost r-paracontact Riemannian manifold M with a structure $\Sigma = (\psi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ is said to be of *P-Sasakian type* if it also satisfies

$$\begin{aligned}(\nabla^*_Z \Psi)(X, Y) &= -\Sigma_\alpha \eta^\alpha(X)[g(Y, Z) - \Sigma_\beta \eta^\beta(Y)\eta^\beta(Z)] \\ &- \Sigma_\alpha \eta^\alpha(Y)[g(X, Z) - \Sigma_\beta \eta^\beta(X)\eta^\beta(Z)]\end{aligned}$$

for all vector fields X, Y and Z on M [8].

The conditions given as above are equivalent respectively to

$$\psi X = \nabla^*_X \xi_\alpha, \quad \alpha \in (r)$$

and

$$(\nabla^*_Y \psi)(X) = -\Sigma_\alpha \eta^\alpha(X)[Y - \eta^\alpha(Y)\xi_\alpha] - [g(X,Y) - \Sigma_\alpha \eta^\alpha(X)\eta^\alpha(Y)]\Sigma_\beta \xi_\beta.$$

In this paper, we study quarter-symmetric non-metric connection in an almost r-paracontact Riemannian manifold. We consider the hypersurfaces and submanifolds of an almost r-paracontact Riemannian manifold endowed with a quarter-symmetric non-metric connection. We also obtain the Gauss and Codazzi equations for hypersurfaces, curvature tensor and the Weingarten equation for submanifolds of an almost r-paracontact Riemannian manifold with respect to the quarter-symmetric non-metric connection.

2. Preliminaries

Let M^{n+1} be an $(n+1)$-dimensional differentiable manifold of class C^∞ and let M^n be the hypersurface in M^{n+1} by the immersion $\tau : M^n \to M^{n+1}$. The differential $d\tau$ of the immersion τ is denoted by B. The vector field X in the tangent space of M^n corresponds to a vector field BX in that of M^{n+1}. Suppose that the enveloping manifold M^{n+1} is an almost r-paracontact Riemannian manifold with metric $\widetilde{g}$. Then the hypersurface M^n is also an almost r-paracontact Riemannian manifold with the induced metric g defined by

$$g(\psi X, Y) = \widetilde{g}(B\psi X, BY),$$

where X and Y are arbitrary vector fields and ψ is a tensor of type (1,1) on M^n. If the Riemannian manifolds M^{n+1} and M^n are both orientable, we can choose a unique vector field N defined along M^n such that

$$\widetilde{g}(B\psi X, N) = 0 \quad \text{and} \quad \widetilde{g}(N,N) = 1$$

for arbitrary vector field X in M^n. We call this vector field as a normal vector field to the hypersurface M^n.

Now, we define a *quarter-symmetric non-metric connection* $\widetilde{\nabla}$ by ([1], [2])

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \widetilde{\dot{\nabla}}_{\widetilde{X}}\widetilde{Y} + \widetilde{\eta}^\alpha(\widetilde{Y})\widetilde{\psi}\widetilde{X} \tag{2.1}$$

for arbitrary vector fields $\widetilde{X}$ and $\widetilde{Y}$ tangents to M^{n+1}, where $\widetilde{\dot{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric $\widetilde{g}$, $\widetilde{\eta}^\alpha$

is a 1-form, $\widetilde{\xi}_\alpha$ is the vector field defined by

$$\widetilde{g}(\widetilde{\xi}_\alpha, \widetilde{X}) = \widetilde{\eta}^\alpha(\widetilde{X})$$

for an arbitrary vector field $\widetilde{X}$ of M^{n+1}. Also

$$\widetilde{g}(\widetilde{\psi}\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{\psi}\widetilde{Y}),$$

where $\widetilde{\psi}$ is a tensor of type (1,1).

Now, suppose that $\Sigma = (\widetilde{\psi}, \widetilde{\xi}_\alpha, \widetilde{\eta}^\alpha, \widetilde{g})_{\alpha\in(r)}$ is an almost r-paracontact Riemannian structure on M^{n+1}, then every vector field $\widetilde{X}$ on M^{n+1} is decomposed as

$$\widetilde{X} = BX + \lambda(X)N,$$

where λ is a 1-form on M^{n+1} and for any vector field X on M^n and normal N. Also we have $b(BX) = b(X)$, $\psi BX = B\psi X$ and $\eta^\alpha(BX) = \eta^\alpha(X)$, where b is a 1-form on M^n.

For each $\alpha\in(r)$, we have [2]

$$\widetilde{\psi}BX = B\psi X + b(X)N \quad \text{and} \quad \psi N = BN' + KN, \tag{2.2}$$

where $b(X) = g(X, N')$, $\widetilde{\xi}_\alpha = B\xi_\alpha + a_\alpha N$ and a_α is defined as

$$a_\alpha = \eta^\alpha(N), \quad \alpha\in(r). \tag{2.3}$$

Now, we define $\widetilde{\eta}^\alpha$ as

$$\widetilde{\eta}^\alpha(BX) = \eta^\alpha(X), \quad \alpha\in(r). \tag{2.4}$$

Theorem 2.1. *The connection induced on the hypersurface of a Riemannian manifold with a quarter-symmetric non-metric connection with respect to the unit normal vector is also a quarter-symmetric non-metric connection.*

Proof. Let $\dot{\nabla}$ be the induced connection from $\widetilde{\dot{\nabla}}$ on the hypersurface with respect to the unit normal vector N, then we have

$$\widetilde{\dot{\nabla}}_{BX}BY = B(\dot{\nabla}_X Y) + h(X,Y)N \tag{2.5}$$

for arbitrary vector fields X and Y on M^n, where h is the second fundamental tensor of the hypersurface M^n. Let ∇ be the connection induced on the hypersurface from $\widetilde{\nabla}$ with respect to the unit normal vector N, then we have

$$\widetilde{\nabla}_{BX}BY = B(\nabla_X Y) + m(X,Y)N \tag{2.6}$$

for arbitrary vector fields X and Y of M^n, m being a tensor field of type (0,2) on the hypersurface M^n.

From equation (2.1), we have

$$\widetilde{\nabla}_{BX}BY = \dot{\widetilde{\nabla}}_{BX}BY + \widetilde{\eta}^{\alpha}(BY)\widetilde{\psi}BX.$$

Using (2.2), (2.4), (2.5) and (2.6) in the above equation, we get

$$\begin{aligned} &B(\nabla_X Y) + m(X,Y)N \\ &= B(\dot{\nabla}_X Y) + h(X,Y)N + \eta^{\alpha}(Y)B\psi X + \eta^{\alpha}(Y)b(X)N. \end{aligned} \tag{2.1}$$

Comparison of the tangential and normal parts in the above equation yield

$$\nabla_X Y = \dot{\nabla}_X Y + \eta^{\alpha}(Y)\psi X$$

and

$$m(X,Y) = h(X,Y) + \eta^{\alpha}(Y)b(X). \tag{2.7}$$

Thus we have

$$\nabla_X Y - \nabla_Y X - [X,Y] = \eta^{\alpha}(Y)\psi X - \eta^{\alpha}(X)\psi Y. \tag{2.8}$$

Hence the connection ∇ induced on M^n is a quarter-symmetric non-metric connection [7]. □

3. Totally geodesic and totally umbilical hypersurfaces

We define $\dot{\nabla}B$ and ∇B respectively by

$$(\dot{\nabla}B)(X,Y) = (\dot{\nabla}_X B)(Y) = \dot{\widetilde{\nabla}}_{BX}BY - B(\dot{\nabla}_X Y)$$

and

$$(\nabla B)(X,Y) = (\nabla_X B)(Y) = \widetilde{\nabla}_{BX}BY - B(\nabla_X Y),$$

where X and Y are arbitrary vector fields on M^n. Then (2.5) and (2.6) take the form respectively

$$(\dot{\nabla}_X B)Y = h(X,Y)N$$

and

$$(\nabla_X B)Y = m(X,Y)N.$$

These are the Gauss equations with respect to the induced connection $\dot{\nabla}$ and ∇, respectively.

Let $X_1, X_2, ..., X_n$ be n-orthonormal vector fields. Then the function

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i, X_i)$$

is called the *mean curvature* of M^n with respect to the Riemannian connection $\dot{\nabla}$ and

$$\frac{1}{n}\sum_{i=1}^{n} m(X_i, X_i)$$

is called the *mean curvature* of M^n with respect to the quarter-symmetric non-metric connection ∇.

From this we have following definitions:

DEFINITION 3.1. The hypersurface M^n is called *totally geodesic* of M^{n+1} with respect to the Riemannian connection $\dot{\nabla}$ if h vanishes.

DEFINITION 3.2. The hypersurface M^n is called *totally umbilical* with respect to the connection $\dot{\nabla}$ if h is proportional to the metric tensor g.

We call M^n is totally geodesic and totally umbilical with respect to the quarter-symmetric non-metric connection ∇ according as the function m vanishes and proportional to the metric g, respectively.

Now we have the following theorems:

THEOREM 3.3. *In order that the mean curvature of the hypersurface M^n with respect to the Riemannian connection $\dot{\nabla}$ concides with that of M^n with respect to the quarter-symmetric non-metric connection ∇ if and only if M^n is invariant.*

Proof. In view of (2.7), we have

$$m(X_i, X_i) = h(X_i, X_i) + \eta^{\alpha}(Y_i)b(X_i).$$

Summing up for $i = 1, 2, ..., n$ and divide by n, we obtain

$$\frac{1}{n}\sum_{i=1}^{n} m(X_i, X_i) = \frac{1}{n}\sum_{i=1}^{n} h(X_i, X_i)$$

if and only if $b(X_i) = 0$, which gives the proof of our theorem. □

THEOREM 3.4. *The hypersurface M^n is totally geodesic with respect to the Riemannian connection ∇ if and only if it is totally geodesic with respect to the quarter-symmetric non-metric connection ∇, provided that M^n is invariant.*

Proof. The proof follows from (2.7) easily. □

4. Gauss, Weingarten and Codazzi equations

In this section we shall obtain the Weingarten equation with respect to the quarter-symmetric non-metric connection $\widetilde{\nabla}$. For the Riemannian connection $\widetilde{\dot{\nabla}}$, these equations are given by

$$\widetilde{\dot{\nabla}}_{BX} N = -BHX \tag{4.1}$$

for any vector field X in M^n, where H is a tensor field of type (1,1) of M^n defined by

$$g(HX, Y) = h(X, Y)$$

from equations (2.1), (2.2) and (2.3) we have

$$\widetilde{\nabla}_{BX} N = \widetilde{\dot{\nabla}}_{BX} N + a_\alpha [B(\psi X) + b(X) N].$$

Using (4.1) we have

$$\widetilde{\nabla}_{BX} N = -BMX + a_\alpha b(X) N, \tag{4.2}$$

where $M = H - a_\alpha \psi$, and X is any vector field in M^n.

Equation (4.2) is the Weingarten equation with respect to the quarter-symmetric non-metric connection.

We shall find the equations of Gauss and Codazzi with respect to the quarter-symmetric non-metric connection. The curvature tensor with respect to the quarter-symmetric non-metric connection $\widetilde{\nabla}$ of M^{n+1} is

$$\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = \widetilde{\nabla}_{\widetilde{X}} \widetilde{\nabla}_{\widetilde{Y}} \widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}} \widetilde{\nabla}_{\widetilde{X}} \widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X}, \widetilde{Y}]} \widetilde{Z}.$$

Putting $\widetilde{X} = BX$, $\widetilde{Y} = BY$ and $\widetilde{Z} = BZ$, we have

$$\widetilde{R}(BX, BY)BZ = \widetilde{\nabla}_{BX} \widetilde{\nabla}_{BY} BZ - \widetilde{\nabla}_{BY} \widetilde{\nabla}_{BX} BZ - \widetilde{\nabla}_{[BX, BY]} BZ.$$

By virtue of (2.6), (2.8), and (4.2), we get

$$\begin{aligned} \widetilde{R}(BX, BY)BZ = {} & B[R(X,Y)Z + m(X,Z)MY - m(Y,Z)MX] \\ & +[(\nabla_X m)(Y,Z) - (\nabla_Y m)(X,Z) + a_\alpha (b(X) - b(Y)) \\ & +m(\eta^\alpha(Y)\psi X - \eta^\alpha(X)\psi Y, Z)]N, \end{aligned} \tag{4.3}$$

where $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ is the curvature tensor of the quarter-symmetric non-metric connection ∇.

Substituting

$$\widetilde{R}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}) = g(\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z}, \widetilde{U})$$

and

$$R(X, Y, Z, U) = g(R(X,Y)Z, U).$$

Then from (4.3), we can easily obtain that

$$\widetilde{R}(BX, BY, BZ, BU) = R(X, Y, Z, U) + m(X, Z)h(Y, U) \tag{4.4}$$
$$-m(Y, Z)h(X, U) + a_\alpha(m(Y, Z)g(\psi X, U) - m(X, Z)g(\psi Y, U))$$

and

$$\widetilde{R}(BX, BY, BZ, N) = (\nabla_X m)(Y, Z) - (\nabla_Y m)(X, Z) \tag{4.5}$$
$$+a_\alpha(b(X) - b(Y)) + m(\eta^\alpha(Y)\psi X - \eta^\alpha(X)\psi Y, Z).$$

Equations (4.4) and (4.5) are the equations of the Gauss and Codazzi with respect to the quarter-symmetric non-metric connection.

5. Submanifolds of co-dimensions 2

Let M^{n+1} be an $(n+1)$-dimensional differentiable manifold of differentiability class C^∞ and let M^{n-1} be an $(n-1)$-dimensional manifold immersed in M^{n+1} by the immersion $\tau : M^{n-1} \to M^{n+1}$. We denote the differentiability $d\tau$ of the immersion τ by B, so that the vector field X in the tangent space of M^{n-1} corresponds to a vector field BX in that of M^{n+1}. Suppose that M^{n+1} is an almost r-paracontact Riemannian manifold with metric tensor $\widetilde{g}$. Then the submanifold M^{n-1} is also an almost r-paracontact Riemannian manifold with metric tensor g such that

$$g(\psi X, Y) = \widetilde{g}(B\psi X, BY)$$

for arbitrary vector fields X, Y in M^{n-1} [3].

Let the manifolds M^{n+1} and M^{n-1} are both orientable such that

$$\widetilde{\psi}BX = B\psi X + a(X)N_1 + b(X)N_2$$
$$\widetilde{g}(B\psi X, N_1) = \widetilde{g}(B\psi X, N_2) = \widetilde{g}(N_1, N_2) = 0$$
$$\text{and} \quad \widetilde{g}(N_1, N_1) = \widetilde{g}(N_2, N_2) = 1$$

for arbitrary vector field X in M^{n-1} [6].

We suppose that the enveloping manifold M^{n+1} admits a quarter-symmetric non-metric connection given by [1]

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \dot{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + \widetilde{\eta}^\alpha(\widetilde{Y})\widetilde{\psi}\widetilde{X}$$

for arbitrary vector field $\widetilde{X}$, $\widetilde{Y}$ in M^{n-1}, $\dot{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric $\widetilde{g}$, $\widetilde{\eta}^\alpha$ is a 1-form. Let us now put

$$\widetilde{\psi}BX = B\psi X + a(X)N_1 + b(X)N_2 \tag{5.1}$$

$$\widetilde{\xi}_\alpha = B\xi_\alpha + a_\alpha N_1 + b_\alpha N_2, \tag{5.2}$$

where $a(X)$ and $b(X)$ are 1-forms on M^{n-1}, ξ_α is a vector field in the tangent space on M^{n-1} and a_α, b_α are functions on M^{n-1} defined by $\eta^\alpha(N_1) = a_\alpha, \quad \eta^\alpha(N_2) = b_\alpha$.

Then we have the following.

THEOREM 5.1. *The connection induced on the submanifold M^{n-1} of co-dimension two of an almost r-paracontact Riemannian manifold M^{n+1} with a quarter-symmetric non-metric connection ∇ is also a quarter-symmetric non-metric connection.*

Proof. Let $\dot{\nabla}$ be the connection induced on the submanifold M^{n-1} from the connection $\widetilde{\dot{\nabla}}$ on the enveloping manifold with respect to unit normal vectors N_1 and N_2, then we have [9]

$$\widetilde{\nabla}_{BX}BY = B(\dot{\nabla}_X Y) + h(X,Y)N_1 + k(X,Y)N_2$$

for arbitrary vector fields X and Y in M^{n-1}, where h and k are the second fundamental tensors of M^{n-1}. Similarly, if ∇ be the connection induced on M^{n-1} from the quarter-symmetric non-metric connection $\widetilde{\nabla}$ on M^{n+1} we have

$$\widetilde{\nabla}_{BX}BY = B(\nabla_X Y) + m(X,Y)N_1 + n(X,Y)N_2, \tag{5.3}$$

where m and n being tensor fields of type (0,2) of the submanifold M^{n-1}.

In view of equation (2.1), we have

$$\widetilde{\nabla}_{BX}BY = \widetilde{\dot{\nabla}}_{BX}BY + \widetilde{\eta}^\alpha(BY)\widetilde{\psi}(BX).$$

Using (5.1), (5.2) and (5.3), we have

$$B(\nabla_X Y)+m(X,Y)N_1+n(X,Y)N_2 = B(\dot{\nabla}_X Y)+h(X,Y)N_1+k(X,Y)N_2$$
$$+\eta^\alpha(Y)(B\psi X + a(X)N_1 + b(X)N_2),$$

where

$$\widetilde{\eta}^\alpha(BY) = \eta^\alpha(Y) \quad \text{and} \quad \widetilde{\psi}(BX) = B\psi X + a(X)N_1 + b(X)N_2.$$

Comparing tangential and normal parts we get

$$\nabla_X Y = \dot{\nabla}_X Y + \eta^\alpha(Y)\psi X,$$

$$m(X,Y) = h(X,Y) + a(X)\eta^\alpha(Y), \tag{5.4$_{(a)}$}$$

$$n(X,Y) = k(X,Y) + b(X)\eta^\alpha(Y). \tag{5.4$_{(b)}$}$$

Thus we have

$$\nabla_X Y - \nabla_Y X - [X,Y] = \eta^\alpha(Y)\psi X - \eta^\alpha(X)\psi Y. \tag{5.5}$$

Hence the connection ∇ induced on M^{n-1} is quarter-symmetric non-metric connection. □

6. Totally geodesic and totally umbilical submanifolds

Let $X_1, X_2, ..., X_{n-1}$ be $(n-1)$-orthonormal vector fields on the submanifold M^{n-1}. Then the function

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}[h(X_i, X_i) + k(X_i, X_i)]$$

is called the *mean curvature* of M^{n-1} with respect to the Riemannian connection $\dot{\nabla}$ and

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}[m(X_i, X_i) + n(X_i, X_i)]$$

is called the *mean curvature* of M^{n-1} with respect to the quarter-symmetric non-metric connection ∇ [6].

From this we have the following definitions.

Definition 6.1. If h and k vanish separately, the submanifold M^{n-1} is called *totally geodesic* with respect to the Riemannian connection $\dot{\nabla}$.

Definition 6.2. The submanifold M^{n-1} is called *totally umbilical* with respect to the connection $\dot{\nabla}$ if h and k are proportional to the metric tensor g.

We call M^{n-1} is *totally geodesic* and *totally umbilical* with respect to the quarter-symmetric non-metric connection ∇ according as the function m and n vanish separately and are proportional to the metric tensor g respectively.

Theorem 6.3. *The mean curvature of M^{n-1} with respect to the Riemannian connection $\dot{\nabla}$ coincides with that of M^{n-1} with respect to the quarter-symmetric non-metric connection ∇ if and only if*

$$\sum_{i=1}^{n-1}[\eta^{\alpha}(Y_i)(a(X_i) + b(X_i))] = 0.$$

Proof. In view of (5.4), we have

$$m(X_i, X_i) + n(X_i, X_i) = h(X_i, X_i) + k(X_i, X_i) + \eta^{\alpha}(Y_i)(a(X_i) + b(X_i)).$$

Summing up for $i = 1, 2, ..., (n-1)$ and then divide it by $2(n-1)$, we get

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}[m(X_i, X_i)+n(X_i, X_i)] = \frac{1}{2(n-1)}\sum_{i=1}^{n-1}[h(X_i, X_i)+k(X_i, X_i)]$$

if and only if $\sum_{i=1}^{n-1}[\eta^\alpha(Y_i)(a(X_i)+b(X_i))] = 0$, which proves our assertion. □

THEOREM 6.4. *The submanifold M^{n-1} is totally geodesic with respect to the Riemannian connection $\dot{\nabla}$ if and only if it is totally geodesic with respect to the quarter-symmetric non-metric connection ∇ provided that $a(X) = 0$ and $b(X) = 0$.*

Proof. The proof follows easily from equations $(5.4)_{(a)}$ and $(5.4)_{(b)}$. □

7. Curvature tensor and Weingarten equations

For the Riemannian connection $\dot{\nabla}$, the Weingarten equations are given by [11]

$$\dot{\widetilde{\nabla}}_{BX}N_1 = -BHX + l(X)N_2, \tag{7.1$_{(a)}$}$$

$$\dot{\widetilde{\nabla}}_{BX}N_2 = -BKX - l(X)N_1, \tag{7.1$_{(b)}$}$$

where H and K are tensor fields of type (1,1) such that $g(HX, Y) = h(X, Y)$ and $g(KX, Y) = k(X, Y)$. Also making use of (2.1), (2.2) and $(7.1)_{(a)}$, we get

$$\widetilde{\nabla}_{BX}N_1 = -B(H - a_\alpha\psi)X + a_\alpha(a(X)N_1 + (b(X)N_2) + l(X)N_2,$$

$$\widetilde{\nabla}_{BX}N_1 = -BM_1X + a_\alpha(a(X)N_1 + (b(X)N_2) + l(X)N_2, \tag{7.2}$$

where

$$M_1X = HX - a_\alpha\psi X.$$

Similarly, from (2.1), (2.2) and $(7.1)_{(b)}$, we get

$$\widetilde{\nabla}_{BX}N_2 = -BM_2X + b_\alpha(a(X)N_1 + (b(X)N_2) - l(X)N_1, \tag{7.3}$$

where

$$M_2X = KX - b_\alpha\psi X.$$

Equations (7.2) and (7.3) are the Weingarten equations with respect to the quarter-symmetric non-metric connection $\widetilde{\nabla}$.

8. Riemannian curvature tensor for quarter-symmetric non-metric connection.

Let $\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z}$ be the Riemannian curvature tensor of the enveloping manifold M^{n+1} with respect to the quarter-symmetric non-metric connection $\widetilde{\nabla}$, then

$$\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}.$$

Putting $\widetilde{X} = BX$, $\widetilde{Y} = BY$ and $\widetilde{Z} = BZ$, we have

$$\widetilde{R}(BX,BY)BZ = \widetilde{\nabla}_{BX}\widetilde{\nabla}_{BY}BZ - \widetilde{\nabla}_{BY}\widetilde{\nabla}_{BX}BZ - \widetilde{\nabla}_{[BX,BY]}BZ.$$

Using (5.3), we get

$$\begin{aligned}\widetilde{R}(BX,BY)BZ &= \widetilde{\nabla}_{BX}(B(\nabla_Y Z) + m(Y,Z)N_1 + n(Y,Z)N_2)\\ &-\widetilde{\nabla}_{BY}(B(\nabla_X Z) + m(X,Z)N_1 + n(X,Z)N_2)\\ &-(B(\nabla_{[X,Y]}Z) + m([X,Y],Z)N_1 + n([X,Y],Z)N_2).\end{aligned}$$

Again using (5.3), (5.5), (7.2) and (7.3), we have

$$\begin{aligned}\widetilde{R}(BX,BY)BZ &= BR(X,Y,Z) + B(m(X,Z)M_1Y - m(Y,Z)M_1X\\ &+n(X,Z)M_2Y - n(Y,Z)M_2X) + m(\eta^\alpha(Y)\psi X - \eta^\alpha(X)\psi Y,Z)N_1\\ &+n(\eta^\alpha(Y)\psi X - \eta^\alpha(X)\psi Y,Z)N_2 + ((\nabla_X m)(Y,Z) - (\nabla_Y m)(X,Z))N_1\\ &+((\nabla_X n)(Y,Z) - (\nabla_Y n)(X,Z))N_2 + l(X)(m(Y,Z)N_2 - n(Y,Z)N_1)\\ &-l(Y)(m(X,Z)N_2 - n(X,Z)N_1) + a_\alpha((a(X)N_1 + b(X)N_2)m(Y,Z)\\ &-(a(Y)N_1 + b(Y)N_2)m(X,Z)) + b_\alpha((a(X)N_1 + b(X)N_2)n(Y,Z)\\ &-(a(Y)N_1 + b(Y)N_2)n(X,Z)),\end{aligned}$$

where $R(X,Y,Z)$ is the Riemannian curvature tensor of the submanifold with respect to the quarter-symmetric non-metric connection ∇.

References

[1] M. Ahmad, J. B. Jun and A. Haseeb, *Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a quarter-symmetric metric connection*, (Accepted) Bulletin of the Korean mathematical Society.

[2] M. Ahmad, C. Ozgur and A. Haseeb, *Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a quarter-symmetric non-metric connection*, (Communicated).

[3] S. Ali and R. Nivas, *On submanifolds of a manifold with a quarter-symmetric connection*, Riv. Mat. Uni. Parma. (6)-3(2000), 11-23.

[4] S. C. Biswas and U. C. De, *Quarter-symmetric connection in an SP-Sasakian manifold*, Comm. Fac. Sci. Univ. Ank. Ser. AL Math. Stat. **46** (1997), no. 1-2, 49-56.

[5] B. Y. Chen, *Geometry of submanifolds*, Marcel Dekker, New York, 1973.

[6] L. S. Das, R. Nivas, S. Ali and M. Ahmad, *Study of Submanifolds immersed in a manifold with quarter-symmetric semi-metric connection*, Tensor, N. S. **65** (2004), 250-260.

[7] S. Golab, *On semi symmetric and quarter-symmetric linear connections*, Tensor, N. S. **29** (1975), 249-254.

[8] I. Mihai and K. Matsumoto, *Submanifolds of an almost r-paracontact manifold of P-Sasakian type*, Tensor, N. S. **48** (1989), no. 2, 136-142.

[9] R. S. Misra, *Almost complex and Almost contact submanifold*, Tensor, N. S. **25** (1972), 419-433.

[10] R. S. Misra and S. N. Pandey, *On quarter-symmetric metric F-connections*, Tensor, N. S. **34** (1980), 1-7.

[11] R. Nivas and G. Verma, *On quarter-symmetric non-metric connection in a Riemannian manifold*, J. Rajasthan Acad. Phys. Sci. **4** (2005), no. 1, 57-68.

[12] S. S. Pujar, *A note on Lorentzian para Sasakian manifold with an LP-contact non-metric quarter-symmetric ψ-connection*, Journal of the Indian Math. Soc. **70** (2003), 1-4.

[13] M. M. Tripathi, *A new connection in a Riemannian manifold*, Int. Elec. J. Geo. **1** (2008), no. 1, 15-24.

*
Department of Mathematics
Integral University
Kursi-Road, Lucknow, 226026, India
E-mail: mobinahmad@rediffmail.com

**
Department of Mathematics
Kookmin University
Seoul 136-702, Republic of Korea
E-mail: jbjun@kookmin.ac.kr

***
Department of Mathematics
Integral University
Kursi-Road, Lucknow, 226026, India