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ON SOME PROPERTIES OF SEMI-INVARIANT SUBMANIFOLDS OF A NEARLY TRANS-SASAKIAN MANIFOLD ADMITTING A QUARTER-SYMMETRIC NON-METRIC CONNECTION

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Abstract. We define a quarter-symmetric non-metric connection in a nearly trans-Sasakian manifold and we consider semi-invariant submanifolds of a nearly trans-Sasakian manifold endowed with a quarter-symmetric non-metric connection. Moreover, we also obtain integrability conditions of the distributions on semi-invariant submanifolds.

1. Introduction

Geometry of submanifolds of Sasakian and Kenmotsu manifolds have been an active area of research (cf. [1], [4] etc.). In 1985, Oubina [16] introduced a new class of almost contact Riemannian manifold known as trans-Sasakian manifold. This class contains α-Sasakian and β-Kenmotsu manifold [12]. Recently, C. Gherghe [10] introduced a nearly trans-Sasakian structure of type (α, β), which generalizes trans-Sasakian structure in the same sense as nearly Sasakian generalizes Sasakian one. A trans-Sasakian structure is always a nearly trans-Sasakian structure. Moreover, a nearly trans-Sasakian structure of type (α, β) is nearly Sasakian [4] or nearly Kenmotsu [1] or nearly cosymplectic [7] accordingly as $\beta = 0$ or $\alpha = 0$ or $\alpha = \beta = 0$.

In 1986, Bejancu [5] introduced the notion of semi-invariant or contact CR-submanifolds as a generalization of invariant and anti-invariant

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submanifolds of an almost contact metric manifold and was followed by several geometers (see [6], [8], [13], [18]).

Let ∇ be a linear connection in an n-dimensional differentiable manifold M.

The torsion tensor T and the curvature tensor R of ∇ are respectively given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In [11], S. Golab introduced the idea of a quarter-symmetric connection. A linear connection is said to be a quarter-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = u(Y)\phi X - u(X)\phi Y,$$

where u is a 1-form and ϕ is a tensor field of type (1, 1). Some properties of quarter symmetric connections are studied in [15, 17].

Moreover, the properties of submanifolds of a Riemannian manifold with quarter-symmetric semi-symmetric connection and quarter-symmetric non-metric connection were studied by L.S. Das et al. in [8, 9]. In [2, 3], M. Ahmad et al. studied some characteristic properties of submanifolds and hypersurfaces of an almost r-paracontact Riemannian manifold endowed with semi-symmetric and quarter-symmetric connections respectively.

In this paper, we study semi-invariant submanifolds of nearly trans-Sasakian manifolds with a quarter-symmetric non-metric connection. The rest of the paper is organized as follows. In Section 2, we give a brief introduction of nearly trans-Sasakian manifold. In Section 3, we recall some necessary details about semi-invarint submanifolds. In Section 4, we derive Nijenhuis tensor for nearly trans-Sasakian manifold with quarter-symmetric non-metric connection. In Section 5, some basic results on nearly trans-Sasakian manifold with quarter-symmetric non-metric connection are obtained. In Sections 6, 7 and 8, integrability of some distributions on nearly trans-Sasakian manifold are discussed.

2. Nearly trans-Sasakian manifold

Let $\bar{M}$ be an almost contact metric manifold [7] with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a (1, 1)-tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi,\ \phi\xi = 0,\ \eta \circ \phi = 0,\ \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y),\ g(X, \xi) = \eta(X) \tag{2.3}$$

for all vector fields X, Y on $\mathrm{T}\bar{M}$. There are two known classes of almost contact metric manifolds, namely Sasakian and Kenmotsu manifolds. Sasakian manifolds are characterized by the tensorial relation

$$(\bar{\bar{\bar{\nabla}}}_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

while Kenmotsu manifolds are given by the tensor equation

$$(\bar{\bar{\bar{\nabla}}}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X.$$

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called a *trans-Sasakian* structure if [16]

$$(\bar{\bar{\bar{\nabla}}}_X \phi)(Y) = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.4}$$

for some smooth functions α and β on $\bar{M}$ and we say that the trans-Sasakian structure is of type (α, β). From the formula (2.4), it follows that [16]

$$\bar{\bar{\bar{\nabla}}}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{2.5}$$

The class $C_6 \oplus C_5$ [14] coincides with the class of trans-Sasakian structures of type (α, β).

We note that trans-Sasakian structures of type $(0, 0)$ are cosymplectic [7], trans-Sasakian structures of tpye $(\alpha, 0)$ are α-Sasakian and $(0, \beta)$ type are β- Kenmotsu [12]. Recently, C. Gherghe [10] introduced a nearly trans-Sasakian structure of type (α, β). An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called a *nearly trans-Sasakian* structure [10] if

$$\begin{aligned}(\bar{\bar{\bar{\nabla}}}_X \phi)Y + (\bar{\bar{\bar{\nabla}}}_Y \phi)X = {} & \alpha(2g(X, Y)\xi - \eta(Y)X - \eta(X)Y) \\ & -\beta(\eta(Y)\phi X + \eta(X)\phi Y).\end{aligned} \tag{2.6}$$

A trans-Sasakian structure is always a nearly trans-Sasakian structure. Moreover a nearly trans-Sasakian structure of type (α, β) is nearly-Sasakian [4], nearly Kenmotsu [1] or nearly cosymplectic [7] accordingly as $\beta = 0$, $\alpha = 1$; or $\alpha = 0$, $\beta = 1$; or $\alpha = 0$, $\beta = 0$ respectively.

A nearly trans-Sasakian structure of type (α, β) will be called α-Sasakian (resp. nearly β-Kenmotsu) if $\beta = 0$(resp. $\alpha = 0$). Thus the structural equations for nearly α-Sasakian, nearly Sasakian, nearly β-Kenmotsu, nearly kenmotsu and nearly cosympletic manifolds are given by

$$(\bar{\bar{\nabla}}_X\phi)Y + (\bar{\bar{\nabla}}_Y\phi)X = \alpha(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y), \tag{2.7}$$

$$(\bar{\bar{\nabla}}_X\phi)Y + (\bar{\bar{\nabla}}_Y\phi)X = 2g(X,Y)\xi - \eta(Y)X - \eta(X)Y, \tag{2.8}$$

$$(\bar{\bar{\nabla}}_X\phi)Y + (\bar{\bar{\nabla}}_Y\phi)X = -\beta(\eta(Y)\phi X + \eta(X)\phi Y), \tag{2.9}$$

$$(\bar{\bar{\nabla}}_X\phi)Y + (\bar{\bar{\nabla}}_Y\phi)X = -\eta(Y)\phi X - \eta(X)\phi Y, \tag{2.10}$$

$$(\bar{\bar{\nabla}}_X\phi)Y + (\bar{\bar{\nabla}}_Y\phi)X = 0 \tag{2.11}$$

respectively.

Now we remark that owing to the existence of a 1-form η, we can define a quarter-symmetric non-metric connection in an almost contact manifold by

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta(Y)\phi X. \tag{2.12}$$

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called a nearly trans-Sasakian structure if (2.6) holds.

Using (2.12) and (2.6), we get

$$\begin{aligned}(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = {} & \alpha(2g(X,Y)\xi + (1-\alpha)(\eta(X)Y + \eta(Y)X)) \\ & -\beta(\eta(Y)\phi X + \eta(X)\phi Y) - 2\eta(X)\eta(Y)\xi,\end{aligned} \tag{2.13}$$

$$\begin{aligned}(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = {} & \alpha(2g(X,Y)\xi + (1-\alpha)(\eta(X)Y + \eta(Y)X)) \\ & -2\eta(X)\eta(Y)\xi,\end{aligned} \tag{2.14}$$

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = 2g(X,Y)\xi - 2\eta(X)\eta(Y)\xi, \tag{2.15}$$

$$\begin{aligned}(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = {} & \eta(X)Y + \eta(Y)X - 2\eta(X)\eta(Y)\xi \\ & -\beta(\eta(Y)\phi X + \eta(X)\phi Y),\end{aligned} \tag{2.16}$$

$$\begin{aligned}(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = {} & \eta(X)Y + \eta(Y)X - 2\eta(X)\eta(Y)\xi \\ & -\eta(Y)\phi X - \eta(X)\phi Y,\end{aligned} \tag{2.17}$$

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = \eta(X)Y + \eta(Y)X - 2\eta(X)\eta(Y)\xi. \tag{2.18}$$

3. Semi-invariant submanifolds

Let M be a submanifold of a Riemannian manifold $\bar{M}$ with Riemannian metric g. Then the Gauss and Weingarten formulae are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y),\ (X,Y \in TM), \tag{3.1}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N + \eta(N)\phi X,\ (N \in T^{\perp}M), \tag{3.2}$$

where $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ are the quarter-symmetric non-metric connection, induced connection and induced normal connections in $\bar{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and h is the second fundamental form related to A by

$$g(h(X,Y),N) = g(A_N X, Y). \tag{3.3}$$

Moreover, if ϕ is a $(1,1)$-tensor field on $\bar{M}$ for $X \in TM$ and $N \in T^{\perp}M$ we have

$$\begin{aligned}(\bar{\nabla}_X\phi)Y &= (\nabla_X P)Y - A_{FY}X - th(X, Y + \eta(FY)PX \\ &\quad +(\nabla_X F)Y + h(X,PY) - fh(X,Y) + \eta(FY)FX,\end{aligned} \tag{3.4}$$

$$\begin{aligned}(\bar{\nabla}_X\phi)N &= (\nabla_X t)N - A_{fN}X - PA_N X + \eta(fN)PX, \\ &\quad +(\nabla_X f)N + h(X,tN) - FA_N X + \eta(fN)FX \\ &\quad +\eta(N)X - \eta(N)\eta(X)\xi,\end{aligned} \tag{3.5}$$

where

$$\phi X = PX + FX,\ (PX \in TM, FX \in T^{\perp}M), \tag{3.6}$$

$$\phi N = tN + fN,\ (tN \in TM, fN \in T^{\perp}M), \tag{3.7}$$

$$(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y,\ (\nabla_X F)Y = \nabla_X{}^{\perp}(FY) - F\nabla_X Y,$$
$$(\nabla_X t)N = \nabla_X(tN) - t\nabla_X{}^{\perp}N,\ (\nabla_X f)N = \nabla_X{}^{\perp}(fN) - f\nabla^{\perp}_X N.$$

The submanifold M is said to be totally geodesic if $h = 0$, minimal if $H = trace(h)/dim(M) = 0$ and totally umbilical if $h(X,Y) = g(X,Y)H$ in $\bar{M}$.

For a distribution D on M, M is said to be *D-totally geodesic* if for all $X,Y \in D$, we have $h(X,Y) = 0$. If for all $X,Y \in D$, we have $h(X,Y) = g(X,Y)K$ for some normal vector K, then M is called *D-totallyumbilical*. For two distributions D and E defined on M, M is said to be (D, E)-*mixed totally geodesic* if for all $X \in D$ and $Y \in E$ we have $h(X, Y) = 0$.

Let D and E be two distributions defined on a manifold M. We say that D is E-*parallel* if for all $X \in E$ and $Y \in D$ we have $\nabla_X Y \in D$. If D is D-parallel then it is called *autoparallel*.

D is called X-*parallel* for some $X \in TM$ if for all $Y \in D$ we have $\nabla_X Y \in D$. D is said to be *parallel* if for all $X \in TM$ and $Y \in D$, $\nabla_X Y \in D$. If a distribution D on M is autoparallel, then it is clearly integrable and by Gauss formula, D is totally geodesic in M. If D is parallel then orthogonal complementary distribution $D^\perp$ is also parallel which implies that D is parallel if and only if $D^\perp$ is parallel. In this case, M is locally the product of the leaves of D and $D^\perp$.

Let M be a submanifold of an almost contact metric manifold. If $\xi \in TM$, then we write $TM = \{\xi\} \oplus \{\xi\}^\perp$, where $\{\xi\}$ is the distribution spanned by ξ and $\{\xi\}^\perp$ is the complementary orthogonal distribution of $\{\xi\}$ in M. Then one gets

$$P\xi = 0 = F\xi, \ \eta o P = 0 = \eta o F, \tag{3.8}$$

$$P^2 + tF = -I + \eta \otimes \xi, \ FP + fF = 0, \tag{3.9}$$

$$f^2 + Ft = -I, \ tf + Pt = 0. \tag{3.10}$$

A submanifold M of an almost contact metric manifold $\bar{M}$ with $\xi \in TM$ is called *semi-invariant* [6] of $\bar{M}$ if there exists two differentiable distributions D^1 and D^0 on M such that

(i) $TM = D^1 \oplus D^0 \oplus \{\xi\}$,
(ii) the distribution D^1 is invariant by ϕ, that is $\phi(D^1) = D^1$ and
(iii) the distribution D^0 is anti $-$ invariant by ϕ, that is $\phi(D^0) \subseteq T^\perp M$.

For $X \in TM$, we can write

$$X = U^1 X + U^0 X + \eta(X)\xi, \tag{3.11}$$

where U^1 and U^0 are the projection operators of TM on D^1 and D^0 respectively. A semi-invariant submanifold of an almost contact metric manifold becomes an invariant submanifold [1,6] (resp. anti-invariant submanifold [1,6]) if $D^0 = \{0\}$ (resp. $D^1 = \{0\}$).

4. Nijenhuis tensor

An almost contact metric manifold is said to be normal [7] if the torsion tensor $N^{(1)}$ vanishes, that is

$$N^{(1)} \equiv [\phi, \phi] + 2d\eta \otimes \xi = 0, \tag{4.1}$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ and d denotes the exterior derivative operator.

In this section, we obtain expression for Nijenhuis tensor $[\phi, \phi]$ of the structure tensor field ϕ given by

$$\begin{aligned} [\phi, \phi](X, Y) = &((\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X) \\ &-\phi((\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X) \end{aligned} \tag{4.2}$$

in a nearly trans-Sasakian manifold. In particular, we derive the expressions for the Nijenhuis tensor $[\phi, \phi]$ in nearly Sasakian manifold and nearly Kenmotsu manifolds.

First, we need the following lemma.

LEMMA 4.1. *In an almost contact metric manifold we have*

$$(\bar{\nabla}_Y\phi)(\phi X) = -(\phi(\bar{\nabla}_Y\phi)X + ((\bar{\nabla}_Y\eta)X)\xi + \eta(X)\bar{\nabla}_Y\xi. \tag{4.3}$$

Proof. For $X, Y \in TM$, we have

$$\begin{aligned} (\bar{\nabla}_Y\phi)(\phi X) &= \bar{\nabla}_Y(\phi^2 X) - \phi(\bar{\nabla}_Y\phi X) + \phi(\phi\bar{\nabla}_Y X) - \phi^2\bar{\nabla}_Y X \\ &= \bar{\nabla}_Y(-X + \eta(X)\xi) - \phi(\bar{\nabla}_Y\phi X) \\ &\quad +\phi(\phi\bar{\nabla}_Y X) - (-\bar{\nabla}_Y X + \eta(\bar{\nabla}_Y X)\xi), \end{aligned}$$

which gives the equation (4.3). □

Now, we prove the following theorem.

THEOREM 4.2. *In a nearly trans-Sasakian manifold with quarter-symmetric non-metric connection, the Nijenhuis tensor $[\phi, \phi]$ of ϕ is given by*

$$\begin{aligned} [\phi, \phi](X, Y) = &4\phi(\bar{\nabla}_Y\phi)X + 2d\eta(X, Y)\xi - \eta(X)\bar{\nabla}_Y\xi + \eta(Y)\bar{\nabla}_X\xi \\ &+(1 - \alpha)(-\eta(Y)\phi X - 3\eta(X)\phi Y) + \alpha(4g(\phi X, Y)\xi \\ &+4\beta\eta(X)\eta(Y)\xi - \beta(-\eta(Y)X + 3\eta(X)Y). \end{aligned} \tag{4.4}$$

Proof. Using Lemma 4.1 and $\eta o\phi = 0$ in (2.6), we get

$$(\bar{\nabla}_{\phi X}\phi)Y = \phi(\bar{\nabla}_Y\phi)X - ((\bar{\nabla}_Y\eta)X)\xi - \eta(X)\bar{\nabla}_Y\xi + \alpha(2g(\phi X,Y)\xi - \eta(Y)\phi X) - (\beta+1)(-\eta(Y)X + \eta(Y)\eta(X)\xi). \tag{4.5}$$

Thus we have

$$\begin{aligned}[\phi,\phi](X,Y) &= ((\bar{\nabla}_{\phi X}\phi)Y + \phi((\bar{\nabla}_Y\phi)X)) - ((\bar{\nabla}_{\phi Y}\phi)X + \phi(\bar{\nabla}_X\phi)Y)\\
&= 2\phi(\bar{\nabla}_Y\phi)X - ((\bar{\nabla}_Y\eta)X)\xi - \eta(X)\bar{\nabla}_Y\xi + \alpha(2g(\phi X,Y)\xi\\
&\quad +((\bar{\nabla}_X\eta)Y)\xi + \eta(Y)\bar{\nabla}_X\xi - \alpha(2g(\phi Y,X)\xi)\\
&\quad -(1-\alpha)(\eta(Y)\phi X) - \beta(\eta(Y)\phi^2X) - 2\phi(\bar{\nabla}_X\phi) + \beta(\eta(X)\phi^2Y)Y\\
&= 2\phi((\bar{\nabla}_Y\phi)X - (\bar{\nabla}_X\phi)Y) + 2d\eta(X,Y)\xi - \eta(X)\bar{\nabla}_Y\xi + \eta(Y)\bar{\nabla}_X\xi\\
&\quad +\beta(\eta(X)\phi^2Y - \eta(Y)\phi^2X) + \alpha(4g(\phi X,Y)\xi\\
&\quad +(1-\alpha)(\eta(Y)\phi X - \eta(X)\phi Y)\\
&= 2\phi((\bar{\nabla}_Y\phi)X + (\bar{\nabla}_Y\phi)X) - 2\alpha\phi(1-\alpha)(\eta(X)Y + \eta(Y)X)\\
&\quad +(1-\alpha)(\eta(Y)\phi X + \eta(X)\phi Y) - 2\beta\phi(\eta(Y)\phi X) + \eta(X)\phi Y\\
&\quad -\eta(X)\bar{\nabla}_Y\xi + \eta(Y)\bar{\nabla}_X\xi - \beta(\eta(X)\phi^2Y - \eta(Y)\phi^2X)\\
&\quad -2\alpha(2g(X,Y)\xi + \alpha(4g(\phi X,Y)\xi + 2d\eta(X,Y)\xi\\
&\quad -2\eta(X)\eta(Y)\xi\\
&= 4\phi(\bar{\nabla}_Y\phi)X - \eta(X)\bar{\nabla}_Y\xi + \eta(Y)\bar{\nabla}_X\xi + 2d\eta(X,Y)\xi\\
&\quad +(1-\alpha)(-\eta(Y)\phi X - 3\eta(X)\phi Y) + \alpha(4g(\phi X,Y)\xi)\\
&\quad +\beta(-\eta(Y)X - 3\eta(X)Y) + 4\beta\eta(X)\eta(Y)\xi,\end{aligned}$$

which implies the equation (4.4). □

From equation (4.4), we get

$$\eta(N^1(X,Y)) = 4d\eta(X,Y) - 4\alpha g(X,\phi Y). \tag{4.6}$$

In particular, if X and Y are perpendicular to ξ, then (4.4) gives

$$[\phi,\phi](X,Y) = 4\phi(\bar{\nabla}_Y\phi)X - 2\eta([X,Y])\xi. \tag{4.7}$$

Corollary 4.3. *In a nearly Sasakian manifold with quarter-symmetric non-metric connection, the Nijenhuis tensor $[\phi,\phi]$ of ϕ is given by*

$$[\phi,\phi](X,Y) = 4\phi(\bar{\nabla}_Y\phi)X + 2d\eta(X,Y)\xi - \eta(X)\bar{\nabla}_Y\xi + \eta(Y)\bar{\nabla}_X\xi + 4g(\phi X,Y)\xi. \tag{4.8}$$

Consequently,

$$\eta(N^1(X,Y)) = 4d\eta(X,Y) - 4g(X,\phi Y). \tag{4.9}$$

In particular, if X and Y are perpendicular to ξ, then

$$[\phi,\phi](X,Y) = 4\phi(\bar{\nabla}_Y\phi)X - 2\eta([X,Y])\xi - 4g(X,\phi Y)\xi. \tag{4.10}$$

Corollary 4.4. *In a nearly Kenmotsu manifold with quarter-symmetric non-metric connection, the Nijenhuis tensor $[\phi,\phi]$ of ϕ is given by*

$$\begin{aligned}[\phi,\phi](X,Y) &= 4\phi(\bar{\nabla}_Y\phi)X + 2d\eta(X,Y)\xi - \eta(X)\bar{\nabla}_Y\xi + \eta(Y)\bar{\nabla}_X\xi \\ &+4\eta(X)\eta(Y)\xi - (\eta(Y)\phi X + 3\eta(X)\phi Y) + (-\eta(Y)X + 3\eta(X)Y).\end{aligned} \tag{4.11}$$

Consequently,

$$\eta(N^1(X,Y)) = 4d\eta(X,Y). \tag{4.12}$$

In particular, if X and Y are perpendicular to ξ, then

$$\begin{aligned}[\phi,\phi](X,Y) &= 4\phi(\bar{\nabla}_Y\phi)X - 2\eta([X,Y])\xi \\ &-(\eta(Y)X - (\eta(Y)\phi X + 3\eta(X)\phi Y).\end{aligned} \tag{4.13}$$

5. Some basic results

Let M be a submanifold of a nearly trans-Sasakian manifold. Using (3.4) and (3.6) in (2.17), we get

$$\begin{aligned}&\alpha(2g(X,Y)\xi + (1-\alpha)(\eta(Y)X + \eta(X)Y) - \beta(\eta(Y)PX \\ &+\eta(Y)FX + \eta(X)PY + \eta(X)FY) - 2\eta(X)\eta(Y)\xi \\ &= (\nabla_X P)Y + (\nabla_Y P)X - A_{FY}X - A_{FX}Y - 2th(X,Y) \\ &\quad -2fh(X,Y) + (\nabla_X F)Y + (\nabla_Y F)X + h(X,PY) + h(PX,Y) \\ &\quad +\eta(FY)PX + \eta(FX)PY + \eta(FY)FX + \eta(FX)FY\end{aligned} \tag{5.1}$$

for any $X, Y \in TM$. Consequently, we have

Proposition 5.1. *Let M be a submanifold of a nearly trans-Sasakian manifold with quarter-symmetric non-metric connetion. Then we have*

$$\begin{aligned}&(\nabla_X P)Y + (\nabla_Y P)X - A_{FY}X - A_{FX}Y - 2th(X,Y) \\ &+\eta(FY)PX + \eta(FX)PY = \alpha(2g(X,Y)\xi \\ &\qquad +(1-\alpha)(\eta(Y)X + \eta(X)Y) \\ &\qquad -\beta(\eta(Y)PX + \eta(X)PY) - 2\eta(X)\eta(Y)\xi\end{aligned} \tag{5.2}$$

and

$$(\nabla_X F)Y + (\nabla_Y F)X + h(X,PY) + h(PX,Y) - 2fh(X,Y) \tag{5.3}$$

$$+\eta(FY)FX + \eta(FX)FY = -\beta(\eta(Y)FX + \eta(X)FY) \\ -\eta(FY)FX - \eta(FX)FY$$

for all $X, Y \in TM$.

Now, we state the following proposition.

PROPOSITION 5.2. *Let* M *be a submanifold of a nearly trans-Sasakian manifold with quarter-symmetric non-metric connetion. Then for all* $X, Y \in TM$ *we get*

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X - \phi[X, Y] \tag{5.4}$$
$$= 2((\nabla_X P)Y - A_{FY}X - th(X, Y) + \eta(FY)PX + \eta(FX)PY) \\ +2((\nabla_X F)Y + h(X, PY) - fh(X, Y) - \eta(FY)FX \\ -\eta(FX)FY - (1-\alpha)(\eta(Y)X + \eta(X)Y - \alpha(2g(X, Y)\xi) \\ +\beta(\eta(Y)PX + \eta(X)PY) + \beta(\eta(Y)FX + \eta(X)FY) \\ +2\eta(X)\eta(Y)\xi.$$

Consequently,

$$P[X, Y] = -\nabla_X PY - \nabla_Y PX + A_{FX}Y + A_{FY}X + 2P\nabla_X Y \\ +2th(X, Y) + 2\eta(FY)PX + 2\eta(FX)PY) \\ +(1-\alpha)(\eta(Y)X + \eta(X)Y) + \alpha(2g(X, Y)\xi \\ -\beta(\eta(Y)PX + \eta(X)PY) - 2\eta(X)\eta(Y)\xi, \tag{5.5}$$

$$F[X, Y] = -\nabla^{\perp}_X FY - \nabla^{\perp}_Y FX - h(X, PY) - h(PX, Y) \\ +2F\nabla_X Y + 2fh(X, Y) + 2\eta(FY)FX + 2\eta(FX)FY \\ -\beta(\eta(Y)FX + \eta(X)FY). \tag{5.6}$$

The proof is straightforward and hence omitted.

PROPOSITION 5.3. *Let* M *be a semi-invariant submanifold of a nearly trans-Sasakian manifold with quarter-symmetric non-metric connection. Then* (P, ξ, η, g) *is a nearly trans-Sasakian structure on the distribution* $D^1 \oplus \{\xi\}$ *if* $th(X, Y) = 0$ *for all* $X, Y \in D^1 \oplus \{\xi\}$.

Proof. From $D^1 \oplus \{\xi\} = Ker(F)$ and (3.9) we have $P^2 = -I + \eta \oplus \xi$ on $D^1 \oplus \{\xi\}$. We also get $P\xi = 0$, $\eta(\xi) = 1$, $\eta o P = 0$. Using $D^1 \oplus \{\xi\} = Ker(F)$ and $th(X, Y) = 0$ in (5.2), we get

$$(\nabla_X P)Y + (\nabla_Y P)X = -\beta(\eta(Y)PX + \eta(X)PY) - \eta(FY)PX \\ -\eta(FX)PY + \alpha(2g(X, Y)\xi + (1-\alpha)(\eta(X)Y - \eta(Y)X) \\ -2\eta(X)\eta(Y)\xi, \tag{5.7}$$

where $X, Y \in D^1 \oplus \{\xi\}$. This completes the proof. □

THEOREM 5.4. *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold with quarter-symmetric non-metric connection. Then we have the followings.*

(i) if $D^0 \oplus \{\xi\}$ is autoparallel, then

$$A_{FX}Y + A_{FY}X + 2th(X,Y) - \eta(FY)PX - \eta(FX)PY = 0, \tag{5.8}$$

$X, Y \in D^0 \oplus \{\xi\}$,

(ii) if $D^1 \oplus \{\xi\}$ is autoparallel, then

$$h(X,PY) + h(PX,Y) + \eta(FY)FX + \eta(FX)FY = 2fh(X,Y), \tag{5.9}$$

$X, Y \in D^1 \oplus \{\xi\}$.

Proof. In view of (5.2) and autoparallelness of $D^0 \oplus \{\xi\}$ we get (1), while in view of (5.3) and appropriateness of $D^1 \oplus \{\xi\}$ we get (2). □

In view of Proposition 5.3 and (ii) in Theorem 5.1, we get

THEOREM 5.5. *Let M be a submanifold of nearly trans-Sasakian manifold with quarter-symmetric non-metric connection with $\xi \in TM$. If M is invariant, then M is nearly trans-Sasakian. Moreover,*

$$h(X,PY) + h(PX,Y) - 2fh(X,Y)\eta(FY)FX + \eta(FX)FY = 0$$

for all $X, Y \in TM$.

6. Integrability of the distribution $D^1 \oplus \{\xi\}$

We begin with a lemma.

LEMMA 6.1. *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold with quarter-symmetric non-metric connection. Then*

$$\begin{aligned} F[X,Y] = &-h(X,PY) - h(PX,Y) + 2F\nabla_X Y + 2fh(X,Y) \\ &+2\eta(FY)FX + 2\eta(FX)FY \end{aligned} \tag{6.1}$$

or equivalently,

$$-h(X,PX) + F\nabla_X X + fh(X,X) + 2\eta(FX)FX = 0 \tag{6.2}$$

for all $X, Y \in D^1 \oplus \{\xi\}$.

Proof. Equation (6.1) follows from $D^1 \oplus \{\xi\} = Ker(F)$ and (5.6) equivalence of (6.1) and $D^1 \oplus \{\xi\} = Ker(F)$. □

We can state the following theorem.

THEOREM 6.2. *The distribution $D^1 \oplus \{\xi\}$ on semi-invariant submanifold of a nearly trans-Sasakian manifold with quarter-symmetric non-metric connection is integrable if and only if*

$$h(X,PY)+h(PX,Y) = 2(F\nabla_X Y + fh(X,Y)) + \eta(FY)FX + \eta(FX)FY \tag{6.3}$$

for all $X,Y \in D^1 \oplus \{\xi\}$.

DEFINITION 6.3. Let M be a Riemannian manifold with the Riemannian connection ∇. A distribution D on M will be called *nearly autoparallel* if for all $X,Y \in D$ we have $(\nabla_X Y + \nabla_Y X) \in D$ or equivalently $\nabla_X X \in D$.

Thus, we have the following flow chart.
Parallel $\Rightarrow$ Autoparallel $\Rightarrow$ Nearly autoparallel,
Parallel $\Rightarrow$ Integrable,
Autoparallel $\Rightarrow$ Integrable,
Nearly autoparallel + Integrable $\Rightarrow$ Autoparallel.

THEOREM 6.4. *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold with quarter-symmetric non-metric connection. Then the following four statements*

(a) the distribution $D^1 \oplus \{\xi\}$ is autoparallel,
(b) $h(X,PY)+h(PX,Y) = 2fh(X,Y) + 2\eta(FX)FY + 2\eta(FY)FX$, *for all $X,Y \in D^1 \oplus \{\xi\}$*,
(c) $h(X,PX) = fh(X,X) + 4\eta(FX)FX,\ X \in D^1 \oplus \{\xi\}$,
(d) the distribution $D^1 \oplus \{\xi\}$ is nearly autoparallel

are related by (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d). In particular, if $D^1 \oplus \{\xi\}$ is integrable, then the above four statement are equivalent.

Let $X,Y \in D^1 \oplus \{\xi\}$. Using (2.1) and (3.6) in (4.1) we get

$$N^1(X,Y) = 2d\eta(X,Y)\xi + [\phi X,\phi Y] - [X,Y] + \eta([X,Y])\xi - P([X,\phi X] + [\phi X,Y]) - F([X,\phi Y] + [\phi X,Y]). \tag{6.4}$$

On the other hand, from equation (4.5) we have

$$(\bar\nabla_{\phi X}\phi)Y = \phi(\bar\nabla_Y\phi)X - ((\bar\nabla_Y\eta)X)\xi - \eta(X)\bar\nabla_Y\xi + \alpha(2g(\phi X,Y)\xi) + (1-\alpha)(\eta(Y)\phi X) - \beta(\eta(Y)\phi^2X)$$

which implies that

$$(\bar\nabla_{\phi X}\phi)Y - (\bar\nabla_{\phi Y}\phi)X = \phi((\bar\nabla_Y\phi)X - (\bar\nabla_X\phi)Y) + 2d\eta(X,Y)\xi \tag{6.5}$$

$$-\eta(X)U^1\nabla_Y\xi - \eta(X)U^0\nabla_Y\xi + \eta(Y)U^1\nabla_X\xi$$
$$+\eta(Y)U^0\nabla_X\xi - \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi)$$
$$+(1-\alpha)(\eta(Y)\phi X - \eta(X)\phi Y)$$
$$-\beta(\eta(X)\phi^2 Y - \eta(Y)\phi^2 X).$$

Next we can easily get

$$\phi(\bar{\nabla}_Y\phi)X = \phi\bar{\nabla}_Y\phi X - \phi^2\bar{\nabla}_Y X \tag{6.6}$$
$$= \phi(\nabla_Y\phi X + h(Y,\phi X)) + \bar{\nabla}_Y X - \eta(\bar{\nabla}_Y X)\xi.$$

So we know

$$\phi((\bar{\nabla}_Y\phi)X - (\bar{\nabla}_X\phi)Y) \tag{6.7}$$
$$= -[X,Y] + \eta([X,Y])\xi + P(\nabla_Y\phi X - \nabla_X\phi Y)$$
$$+F(\nabla_Y\phi X - \nabla_X\phi Y) + \phi(h(Y,\phi X) - h(X,\phi Y)).$$

In view of (6.5) and (6.7), we get

$$N^1(X,Y) = -2[X,Y] + 2P(\nabla_Y\phi X - \nabla_X\phi Y) \tag{6.8}$$
$$+2F(\nabla_Y\phi X - \nabla_X\phi Y) + 2\phi(h(Y,\phi X) - h(X,\phi Y))$$
$$-\eta(X)U^1\nabla_Y\xi - \eta(X)U^0\nabla_Y\xi + \eta(Y)U^1\nabla_X\xi$$
$$+\eta(Y)U^0\nabla_X\xi - \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi)$$
$$+4d\eta(X,Y)\xi + (1-\alpha)(\eta(Y)\phi X - \eta(X)\phi Y)$$
$$-\beta(\eta(Y)X - \eta(X)Y) + 2\eta([X,Y])\xi - \beta(2\eta(X)\eta(Y)\xi).$$

THEOREM 6.5. *The distribution $D^1 \oplus \{\xi\}$ is integrable on a semi-invariant submanifold M of nearly trans-Sasakian manifold with quarter-symmetric non-metric connection if and only if*

$$N^1(X,Y) \in D^1 \oplus \{\xi\}, \tag{6.9}$$

$$2(h(Y,\phi X) - h(X,\phi Y)) = \eta(X)(\phi U^0\nabla_Y\xi + (1-\alpha)U^0 Y \tag{6.10}$$
$$+\beta\phi U^0 Y + fh(Y,\xi))$$
$$-\eta(Y)(\phi U^0\nabla_X\xi + (1-\alpha)U^0 X + \beta\phi U^0 X + fh(X,\xi))$$

for all $X, Y \in D^1 \oplus \{\xi\}$.

Proof. Let $X, Y \in D^1 \oplus \{\xi\}$. If $D^1 \oplus \{\xi\}$ is integrable, then (6.9) is true and we get from (6.8)

$$\begin{aligned}&2F(\nabla_Y \phi X - \nabla_X \phi Y) + 2\phi(h(Y, \phi X) - h(X, \phi Y))\\&+\eta(Y)U^0\nabla_X\xi - \eta(X)U^0\nabla_Y\xi + \eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)\\&+(1-\alpha)(\eta(Y)FX - \eta(X)FY) - \beta(\eta(Y)X - \eta(X)Y\\&+2\eta(X)\eta(Y)\xi) = 0.\end{aligned}$$

Applying ϕ to the above equation, we have

$$\begin{aligned}&-2U^0(\nabla_Y \phi X - \nabla_X \phi Y) - 2h(Y, \phi X) - h(X, \phi Y)\\&+\eta(Y)\phi U^0\nabla_X\xi - \eta(X)\phi U^0\nabla_Y\xi + \eta(Y)th(X,\xi)\\&+\eta(Y)fh(X,\xi) - \eta(X)th(Y,\xi) - \eta(X)fh(Y,\xi)\\&+(1-\alpha)(\eta(Y)U^0X + \eta(X)U^0Y) + \beta\phi(\eta(X)U^0Y + \eta(Y)U^0X) = 0.\end{aligned}$$

Hence taking the normal part, we get (6.10).

Conversely, let (6.9) and (6.10) be true. Using (6.10) in (6.8) we get

$$\begin{aligned}&-2U^0[X,Y] + 2F(\nabla_Y \phi X - \nabla_X \phi Y) + 2\phi h(Y, \phi X) - h(X, \phi Y)\\&+\eta(Y)U^0\nabla_X\xi - \eta(X)U^0\nabla_Y\xi + \eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)\\&+(1-\alpha)(\eta(Y)FX - \eta(X)FY) - \beta(\eta(X)\phi^2Y - \eta(Y)\phi^2X) = 0.\end{aligned}$$

Applying ϕ to the above equation and using (6.10), we get $\phi U^0[X, Y] = 0$, from which we get $U^0[X, Y] = 0$. Hence $D^1 \oplus \{\xi\}$ is integrable. □

If $\bar{M}$ is a trans-Sasakian manifold, then it is known that $h(X, \xi) = 0$ and $U^0\nabla_X\xi = 0$ for all $X \in D^1 \oplus \{\xi\}$. Hence in view of the previous theorem, we have the following.

COROLLARY 6.6. *If M is a semi-invariant submanifold of a trans-Sasakian manifold with quarter-symmetric non-metric connection, then the distribution $D^1 \oplus \{\xi\}$ is integrable if and only if $h(X, \phi Y) = h(Y, \phi X)$ for all $X, Y \in D^1 \oplus \{\xi\}$.*

7. Integrability of the distribution $D^0 \oplus \{\xi\}$

LEMMA 7.1. *Let M be a semi-invariant submanifold of a trans-Sasakian manifold with quarter-symmetric non-metric connection. Then*

$$3(A_{FX}Y - A_{FY}X) = P[X,Y] - \beta(\eta(Y)PX + \eta(X)PY). \tag{7.1}$$

Proof. For $X, Y \in D^0 \oplus \{\xi\}$ and $Z \in TM$, we have

$$-A_{\phi X}Z + \nabla^{\perp}_Z \phi X = \bar{\nabla}_Z \phi X = (\bar{\nabla}_Z \phi)X + \phi(\bar{\nabla}_Z X).$$

Using equation (2.13) in above, we have

$$\begin{aligned} -A_{\phi X}Z + \nabla^{\perp}_Z \phi X = &-(\bar{\nabla}_X \phi)Z + \alpha(2g(X,Z)\xi \\ &+(1-\alpha)\eta(X)Z - \eta(Z)X) - \beta(\eta(X)\phi Z + \eta(Z)\phi X) \\ &-2\eta(X)\eta(Z)\xi + \phi\bar{\nabla}_Z X + \phi h(Z,X), \end{aligned}$$

so that

$$\begin{aligned} \phi h(Z,X) = &-A_{\phi X}Z + \nabla^{\perp}_Z \phi X + (\bar{\nabla}_X \phi)Z - \alpha(2g(X,Z)\xi \\ &+(1-\alpha)\eta(X)Z - \eta(Z)X) - \beta(\eta(Z)\phi X + \eta(X)\phi Z) \\ &-2\eta(X)\eta(Y)\xi - \phi\bar{\nabla}_Z X \end{aligned}$$

and hence, we have

$$\begin{aligned} g(\phi h(Z,X),Y) &= -g(A_{\phi X}Z,Y) + g((\bar{\nabla}_X \phi)Z,Y) \\ &= -g(A_{\phi X}Y,Z) - g((\bar{\nabla}_X \phi)Y,Z). \end{aligned}$$

On the other hand, we know

$$g(\phi h(Z,X),Y) = -g(h(Z,X),\phi Y) = -g(A_{\phi Y}X,Z).$$

Thus from the above two relations, we get

$$g(A_{\phi Y}X,Z) = g(A_{\phi X}Y,Z) + g((\bar{\nabla}_X \phi)Y,Z). \tag{7.2}$$

For $X, Y \in D^0 \oplus \{\xi\}$, we calculate $(\bar{\nabla}_X \phi)Y$ as follows. In view of

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X$$

and

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X + \phi[X,Y],$$

we have

$$(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X - \phi[X,Y],$$

which in view of (2.13) gives

$$\begin{aligned} (\bar{\nabla}_X \phi)Y = &\frac{1}{2}(A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X - \phi[X,Y] \\ &+\frac{\alpha}{2}(2g(X,Y)\xi) + \frac{(1-\alpha)}{2}(\eta(X)Y + \eta(Y)X) \\ &-\frac{\beta}{2}(\eta(X)\phi Y + \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi). \end{aligned} \tag{7.3}$$

Now using (7.3) in (7.2), we get (7.1). □

In view of $Ker(P) = D^0 \oplus \{\xi\}$, this leads to the following.

THEOREM 7.2. *Let M be semi-invariant submanifold of a nearly trans-Sasakian manifold with quarter-symmetric non-metric connection. Then the distribution $D^0 \oplus \{\xi\}$ is integrable if and only if*

$$A_{FX}Y = A_{FY}X, \;\; X, Y \in D^0 \oplus \{\xi\}.$$

Using (2.4) in (7.2) for $X, Y \in D^0 \oplus \{\xi\}$, we get $A_{FX}Y = A_{FY}X$. Hence, in view of the above theorem, we get the following.

COROLLARY 7.3. *Let M be a semi-invariant submanifold of a trans-Sasakian manifold with quarter-symmetric non-metric connection. Then the distribution $D^0 \oplus \{\xi\}$ is integrable.*

8. Integrability of the distribution D^0

We calculated the torsion tensor $N^1(Y, X)$ for $X, Y \in D^0$, it can be verified that

$$\begin{aligned}\phi((\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X) &= \phi(A_{\phi X}Y - A_{\phi Y}X) \\ &\quad + \phi(\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X) + [X, Y] - \eta([X, Y])\xi,\end{aligned} \tag{8.1}$$

$$\begin{aligned}(\bar{\nabla}_{\phi Y} \phi)X - (\bar{\nabla}_{\phi X} \phi)Y) &= [X, Y] + \phi(A_{\phi X}Y - A_{\phi Y}X) \\ &\quad + \phi(\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X).\end{aligned} \tag{8.2}$$

Using (8.1), (8.2) and (7.1), we get for $X, Y \in D^0$

$$N^1(Y, X) = \frac{8}{3}[X, Y] + \frac{2}{3}\phi(\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X) + \frac{8}{3} d\eta(X, Y)\xi. \tag{8.3}$$

THEOREM 8.1. *The distribution D^0 is integrable on a semi-nvariant submanifold M of a nearly trans-Sasakian manifold with quarter-symmetric non-metric connection if and only if*

$$N^1(Y, X) \in D^0 \oplus \bar{D}^1, \tag{8.4}$$

$$A_{FX}Y = A_{FY}X, \tag{8.5}$$

for all $X, Y \in D^0$.

Proof. If D^0 is integrable, then in view of (8.2) and (8.3) the relation (8.4) and (8.5) follow easily. Conversely, let $X, Y \in D^0$ and let the relation (8.4) and (8.5) be true. Then in view of (8.2) we get $P[X, Y] = 0$ and in view of (8.3) we get

$$g(\xi, N^1(Y, X)) = g(\xi, 2[Y, X]) = 0.$$

Thus $[X, Y] \in D^0$. □

9. Non-integrability of the distribution D^1

THEOREM 9.1. *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold with quarter-symmetric non-metric connection with $\alpha \neq 0$. Then the non-zero invariant distribution D^1 is not integrable.*

Proof. If D^1 is integrable, then it follows that $d\eta(X,Y) = 0$ and $[\phi,\phi](X,Y) \in D^1$ for $X,Y \in D^1$. Therefore, for $X \in D^1$, (4.6) gives

$$\eta([\phi,\phi](X,PX) + 2d\eta(X,PX)\xi) = 0,$$
$$\eta(N^1(X,PX)) = 4\alpha g(\phi X,PX) = 4\alpha g(PX,PX),$$

which is a contradiction. □

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