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BAER SPECIAL RINGS AND REVERSIBILITY

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Abstract. In this paper, we apply some properties of reversible rings, Baerness of fixed rings, skew group rings and Morita Context rings to get conditions that shows fixed rings, skew group rings and Morita Context rings are reversible. Moreover, we investigate conditions in which Baer rings are reversible and reversible rings are Baer.

1. Introduction

Throughout this paper all rings are associative rings with identity unless otherwise stated. Let R be a ring. We denote $S_r(R)$(resp. $S_l(R)$ by the set of right(resp. left)semi-central idempotents in R. For a nonempty subset X of R, $r_R(X)$(resp. $l_R(X)$) will be denoted by the right(resp. left)annihilator of X in R. In 1990, Habeb studied zero commutative ring in [5]. A ring R is called zero commutative, if $ab = 0$ implies $ba = 0$ for any $a, b \in R$. In 1999[4] used the terminology "*reversible ring*" instead of "*zero commutative* ". Obviously, a commutative ring is reversible ring, but converse is not true. In fact every reversible ring is semi-commutative but converse is not true(for more details see [12]). Moreover, A ring R is called right (resp. left)*symmetric ring*, if $rst = 0$ implies $rts = 0$ (resp. $srt = 0$), for any $r, s, t \in R$. It is easy to check that reduced ring is symmetric ring,a symmetric ring with identity is a reversible and a reversible ring is semi-commutative. In 2002, Marks [16] studied conditions in which a group ring becomes reversible, and studied some relationships of among symmetric rings, reduced rings and reversible rings. Baer ring is one of the classic rings and it is applied widely in the field of C*-algebra, Von Neumann algebra and Coding

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Theory. A ring R is called *Baer ring* if a right annihilator of every nonempty subset of R is generated, as a right ideal, by an idempotent of R. The definition is left-right symmetric. In[1],[2] and [8] it is proved Baer ring is quasi-Baer ring, but Converse does not hold. On the basis of conclusion above, Birkenmeier [2] proved that biregular rings and quasi-Baer rings are p.q.-Baer rings. Also he proved that quasi-Baer ring and p.q.-Baer ring are closed under direct product, and right p.q.-Baer rings have Morita invariant property. Furthermore, Jin and Zhao [9] have studied (quasi-) Baerness of skew group rings and fixed rings, and proved that if R is simple ring with identity, G is an outer group of ring automorphism of R, then a skew group ring $R * G$ is Baer ring. similarly, if R is Artin simple ring with identity, G is an outer group of ring automorphism of R, then a fixed ring R^G is Baer ring. Through the way of factorizating Morita Context ring, Jin [8] found conditions in which Morita Context ring becomes a (quasi-)Baer ring. Morita [10] introduce a Morita Context ring and studied its structure. Morita Context ring is generalized matrix ring $(R, V, W, S, \psi, \varphi)$ with six in one algebra structure. It is proved that all of 2×2 matrix ring is Morita Context ring under addition and multiplication of matrices. Montgomery [18] used Morita Context theory to study properties of Morita Context ring with two zero-module homomorphism. Wang [20]used a counterexample to prove Baer ring has not Morita invariant property and used Morita Context theory to study Baerness, (quasi-)Baerness and right quasi Baerness of 2×2 Morita Context ring with zero-module homomorphism and then extended it to 3×3 Morita Context ring. But the reversibility of fixed ring, skew group ring, Morita Context ring and Baer ring have not been studied. So we are going to study the reversibility of ring, and Baerness of fixed rings, skew group rings and Morita Context rings. Furthermore we obtaine conditions in which fixed ring, skew group ring and Morita Context ring become reversible and conditions in which Baer ring and reversible ring replace each other.

2. Preliminaries

Definition 2.1. [4] A ring R is called a *reversible*, if $ab = 0$ implies $ba = 0$ for any $a, b \in R$.

Evidently, commutative ring is a reversible, and a ring that have no zero divisor is a reversible. In fact, if R is a division ring and $ab = 0$, then $a = 0$ or $b = 0$, so $ba = 0$, for any $a, b \in R$.

PROPOSITION 2.2. [15] *Every reduced ring is a reversible ring, but the converse does not hold.*

EXAMPLE 2.3. *Let R is a reduced ring and* $S = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| a, b \in R \right\}$, *then S is a reversible ring, but S is not a reduced ring. In fact, for any* $\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} \in S$, if

$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1a_2 \\ 0 & a_1a_2 \end{pmatrix} = 0$$

then $a_1a_2 = 0$, $a_1b_2 + b_1a_2 = 0$. since R is a reduced ring, R is a reversible ring, so $a_2a_1 = 0$. And $a_2a_1b_2 + a_2b_1a_2 = 0$, so $a_2b_1a_2 = 0$. That is $a_2b_1a_2b_1 = 0$ and $a_2b_1 = 0$, thus $b_1a_2 = 0$ consequently we have $a_1b_2 = 0$ and so $b_2a_1 = 0$. Then

$$\begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} = \begin{pmatrix} a_2a_1 & a_2b_1 + b_2a_1 \\ 0 & a_2a_1 \end{pmatrix} = 0.$$

And S is a reversible ring. Otherwise, take a nonzero element $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ *in S, since* $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^2 = 0$, *$S$ is not a reduced ring.*

PROPOSITION 2.4. *Let R be a reduced ring. Then the polynomial ring $R[x]$ is a reversible ring.*

Proof. For any $\sum\limits_{i=1}^{m} a_ix^i$, $\sum\limits_{i=1}^{n} b_jx^j \in R[x]$, if $(\sum\limits_{i=1}^{m} a_ix^i)(\sum\limits_{i=1}^{n} b_jx^j) = \sum\limits_{k=1}^{m+n} c_kx^k = 0$, then $c_k = \sum\limits_{i+j=k} a_ib_j = 0 \quad (k = 0, 1, 2, \cdots m+n)$ where $c_k = \sum\limits_{i+j=k} a_ib_j$. That is

$$\begin{aligned} c_0 &= a_0b_0 = 0 \\ c_1 &= a_0b_1 + a_1b_0 = 0 \\ c_2 &= a_0b_2 + a_1b_1 + a_2b_0 = 0 \\ &\vdots \\ c_{m+n} &= a_0b_{m+n} + a_1b_{m+n-1} + \cdots + a_{m+n}b_0 = 0 \end{aligned}$$

since R is a reduced ring, so R is a reversible ring and $b_0a_0 = 0$. Because $c_1a_0 = a_0b_1a_0 + a_1b_0a_0 = 0$ we have $a_0b_1a_0 = 0$ and $(a_0b_1)^2 = 0$. since R is a reduced ring we get $a_0b_1 = 0$ and $a_1b_0 = 0$. Hence

$$a_0b_1 + a_1b_0 = b_1a_0 + b_0a_1 = 0.$$

Similarly we obtain $a_ib_j = b_ja_i = 0$ and so $c'_k = \sum_{i+j=k} b_ja_i = 0$. Then $(\sum_{i=1}^{n} b_jx^j)(\sum_{i=1}^{m} a_ix^i) = \sum_{k=1}^{m+n} c'_kx^k = 0$ and hence $R[x]$ is a reversible ring.

DEFINITION 2.5. [12] A ring R is called a *semi-commutative* if $ab = 0$ implies $aRb = 0$, for any $a, b \in R$.

PROPOSITION 2.6. [12] *Every reversible ring is a semi-commutative ring, but the reverse is not true.*

Proof. Suppose that R is a reversible ring. Then $ab = 0$ implies $ba = 0$ for any $a, b \in R$. So $(ba)r = 0$, for any $r \in R$ and $(ar)b = 0$. Thus $aRb = 0$, hence R is a semi-commutative ring. □

EXAMPLE 2.7. [12] *Let R is a reduced ring. Then ring* $S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in R \right\}$ *is a semi-commutative but it is not reversible.*

PROPOSITION 2.8. [3] *Let R is a ring, for an idempotent $e \in R$ the following conditions are equivalent:*

(1) $e \in S_l(R)$.
(2) $1 - e \in S_r(R)$.
(3) *for any $x \in R$, there have $xe = exe$.*
(4) $(1 - e)Re = 0$.
(5) *eR is an ideal of R.*
(6) *$eR(1 - e)$ is an ideal of R, and $eR = eR(1 - e) \oplus Re$.*

3. Main results

DEFINITION 3.1. [7] A ring R is called an *Abel ring* if every idempotent of ring is a central idempotent.

It is easy to verify that the commutative ring is a Abel ring, Suppose F is a field, then a ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| a, b \in F \right\}$ is also a Abel ring.

PROPOSITION 3.2. [20] *The formal matrix ring $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ is an Abel ring if and only if ring A, B are Abel rings and $M = 0$.*

Lemma 3.3. [11] *A ring R is a semi-commutative if and only if the following three equivalent statements hold:*

(1) *Any right annihilator over R is an ideal of R.*
(2) *Any left annihilator over R is an ideal of R.*
(3) *For any $a, b \in R$ $ab = 0$ implies $aRb = 0$.*

Theorem 3.4. *Let R be a reversible ring and e is an idempotent of ring R, then R is a Baer ring.*

Proof. Let R be a reversible ring and X be a nonempty subgroup of ring R, then R is a semi-commutative ring by Proposition 2.6. By Lemma 3.3, $r_R(X) = eR$, for $e^2 = e \in R$. So R is a Baer ring by Proposition 2.8. □

Theorem 3.5. *Let R is a Abel ring, then the following conditions are equivalent:*

(1) *R is a reversible ring.*
(2) *R is a Baer ring.*

Proof. (1)⇒(2) It is trivial by Theorem 3.4.
(2)⇒(1) For any $a \in R$, if $a^2 = 0$, then $a \in r_R(a)$, since R is a Baer ring, so there exists $e = e^2 \in R$, such that $a \in r_R(a) = eR$, so there exists $x \in R$, such that $a = ex$. Since R is a Abel ring, then $a = ex = xe$, for R is a ring with identity, then $e \in eR$, so $0 = ae = exe = e^2x = ex$. Hence $a = 0$, so R is a reduced ring, thus R is a reversible ring by Proposition 2.2. □

Note that the reverse of Theorem 3.4 is not hold and the condition of R be a Abel ring is necessary by following example.

Example 3.6. *Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in F \right\}$ is a ring, F is a field, then all idempotents of R are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, for any $x, y \in F$, the element of R can be expressed by the form of*
$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$
for any $a, b, c \in F$ and $a \neq 0, b \neq 0, c \neq 0$, so

$$r_R\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} R, \quad r_R\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -a^{-1}b \\ 0 & 1 \end{pmatrix} R,$$

$$r_R\left(\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} R, \quad r_R\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} R,$$

$$r_R\left(\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R, \quad r_R\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R,$$

$$r_R\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & c \end{array}\right)\right) = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right)R.$$

Let X be a nonempty subset of ring R, then $r_R(X) = \bigcap\limits_{x_i \in X} r_R(x_i)$, so

$$r_R(x_i \cap x_j) = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right)R, i \neq j = 1, 2, 3, 4, 5, 6, 7.$$

So R is a Baer ring.

Since $\left(\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array}\right)\left(\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right) = 0$, but $\left(\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right)\left(\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array}\right) \neq 0$, hence R is not a reversible ring.

Because for any $r \in R$ and an idempotent $\left(\begin{array}{cc} 1 & x \\ 0 & 0 \end{array}\right)$ of R, $\left(\begin{array}{cc} 1 & x \\ 0 & 0 \end{array}\right)r$ $\neq r\left(\begin{array}{cc} 1 & x \\ 0 & 0 \end{array}\right)$, so $\left(\begin{array}{cc} 1 & x \\ 0 & 0 \end{array}\right)$ is not a central idempotent, hence is not a Abel ring.

Definition 3.7. Let $Aut(R)$ denote group of ring automorphism of R, and G be a subgroup of $Aut(R)$. We use R^G to denote a *fixed ring* of R under G, i.e.,

$$R^G = \{r \in R | r^g = r, \forall g \in G\},$$

where $\varphi : G \to Aut(R)$ is group homomorphism and for any $g \in G$ and $r \in R$, define $r^g = \varphi(g)(r)$.

It can be shown that R^G is a subring of R. Suppose $G = \{id\}$, $id \in Aut(R)$, then $G \leq Aut(R)$ and $R^G = R$.

Definition 3.8. Let R be a ring with identity, $U(R)$ is an unit set of R, for $a \in U(R)$ and $r \in R$ a mapping σ_a:$R \to R$ is defined by $\sigma_a(r) = ara^{-1}$ for $r \in R$, then σ_a is a automorphism of R.

Let $Int(R) = \{\sigma_a | \sigma_a \in Aut(R)\}$, then $Int(R)$ is a subgroup of $Aut(R)$ and $Int(R)$ is called a *group of Inner automorphism.*

If identity mapping is only inner automorphism in G, then a subgroup G of $Aut(R)$ is called an *Outer automorphism group.*

Example 3.9. *Let R be a ring, $G = Int(R)$, $\varphi : Int(R) \to Aut(R)$, $g \mapsto \varphi(g) = g$, where φ is an identity group of endomorphism, then $R^G = \{r \in R | r^g = r, \forall g \in G\} = \left\{r \in R | ara^{-1} = r, a \in U(R)\right\}$.*

In fact, since $\forall g \in Int(R) \leq Aut(R), r \in R$, $a \in U(R)$, since φ is an identity group of endomorphism, so $r = r^g = \varphi(g)(r) = g(r) = ara^{-1}$, for $\forall r \in R^G$, thus $R^G = \left\{r \in R | ara^{-1} = r, a \in U(R)\right\}$.

DEFINITION 3.10. Let R be a ring, G is a subgroup of $Aut(R)$, $\varphi : G \to Aut(R)$ is a group homomorphism. Let $R*G = \{ \sum_{g\in G} r_g g | r_g \in R,$ for only finite $r_g \neq 0$ $\}$, for any $\sum_{g\in G} r_g g$, $\sum_{g\in G} r'_g g$, $\sum_{h\in G} r_h h \in R*G$, define

$$\sum_{g\in G} r_g g + \sum_{g\in G} r'_g g = \sum_{g\in G} (r_g + r'_g) g,$$

$$\left(\sum_{g\in G} r_g g\right)\left(\sum_{h\in G} r_h h\right) = \sum_{g\in G}\sum_{h\in G} (r_g r_h^{g^{-1}}) gh$$

there have $r_h^{g^{-1}} = \varphi(g^{-1})(r_h)$, then $R*G$ is a ring, it is called a *skew group ring.*

LEMMA 3.11. [9] *Let R be a simple ring, G be an Outer group of ring automorphism of R , then $R*G$ is a simple ring; If R is a Artin ring, then $R*G$ is a Artin simple ring.*

LEMMA 3.12. [9] *Every semisimple ring is a Baer ring and every semisimple module is a Baer module.*

PROPOSITION 3.13. [9] *Let R be a Artin simple ring, G be an Outer group of ring automorphism of R, then R^G is a Baer ring.*

THEOREM 3.14. *Let R be a reversible ring, G be a subgroup of $Aut(R)$, then R^G is a reversible ring.*

Proof. For any $a, b \in R^G, g \in G$, there exists $a^g = a, b^g = b$, if $ab = 0$ in R^G, then $ab = 0$ in R, since R is a reversible and R^G is a subring of R. So $ba = b^g a^g = \varphi(g)(b)\varphi(g)(a) = \varphi(g)(ba) = \varphi(g)(0) = 0$ in R^G, then R^G is a reversible ring. □

We can know that the reverse of this theorem is not hold by following example.

EXAMPLE 3.15. *Let* $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{Z}_2 \right\}$, *then $R^{Int(R)}$ is a reversible ring, but R is not a reversible ring.*

As a matter of fact, since $U(R) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$, so $Int(R) = \left\{ f_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}, f_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)} \right\}$. For every $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$, since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

So $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, thus $a+b+c=b$, that is $a=c$.

Since $\left\{ r \in R \,\middle|\, f_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}(r) = r \right\} = R$, so

$$\begin{aligned} R^{Int(R)} &= \{r \in R | r^g = r, \forall \text{ g} \in Int(R)\} \\ &= \left\{ r \in R \,\middle|\, f_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}(r) = r, f_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}(r) = r \right\} \\ &= \left\{ r \in R \,\middle|\, f_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}(r) = r \right\} \cap \left\{ r \in R \,\middle|\, f_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}(r) = r \right\} \\ &= R \cap \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{Z}_2 \right\} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{Z}_2 \right\}. \end{aligned}$$

So $R^{Int(R)}$ is a reversible ring. But R is not a reversible ring. Because $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

THEOREM 3.16. *Let R be a reversible Artin simple ring, G is an Outer group of ring automorphism of R, then R^G is a Baer ring, Moreover, R^G is a reversible ring.*

Proof. Since R is a Artin simple ring, G is an Outer group of ring automorphism of R, so R^G is a Baer ring by Proposition 3.13, since R is a reversible ring, $G \leq Aut(R)$. So R^G is a reversible ring by Theorem 3.14. □

PROPOSITION 3.17. *Let R be a ring with identity and G is a group of ring automorphism of R, then G is a subgroup of $U(R * G)$, under the meaning of isomorphism.*

Proof. Let $H = \{1_R\, g \mid \forall g \in G\}$, then $H \leq U(R * G)$. In fact, since $1_R\, 1_G \in H$, so H is a nonempty subset, $(1_R\, g_1)(1_R\, g_2)^{-1} = (1_R\, g_1)(1_R\, g_2^{-1}) = 1_R\, 1_R^{g_1^{-1}}\, g_1\, g_2^{-1} = 1_R\, (g_1 \cdot g_2^{-1}) \in H$, for $\forall\, 1_R\, g_1, 1_R\, g_2 \in H$. So H is a subgroup of $U(R*G)$. Let $f : G \to H$, $g \mapsto 1_R \cdot g$, that is $f(g) = 1_R \cdot g$, so f is bijection by $G \to H$. For $\forall g_1, g_2 \in G$, $f(g_1 g_2) = 1_R(g_1 g_2) = 1_R\, 1_R\, (g_1 g_2) = 1_R\, 1_R^{g_1^{-1}}\, (g_1 g_2) = (1_R\, g_1)\, (1_R\, g_2) = f(g_1)\, f(g_2)$. So f is isomorphism by $G \to H$. Hence G is a subgroup of $U(R * G)$. □

THEOREM 3.18. *Let R be a simple ring, G is an Outer group of ring automorphism of R, then $R * G$ is a Baer ring.*

Proof. Suppose R be a simple ring, G is an Outer group of ring automorphism of R, then $R * G$ is a simple ring by Lemma 3.11, since R is a ring with identity, so R is a semisimple ring, thus $R*G$ is a Baer ring by Lemma 3.12. □

Definition 3.19. [8] Let R be a semiprime ring. For $g \in Aut(R)$, Let $\phi_g = \{x \in Q^r(R) | xr^g = rx$, for each $r \in R\}$, where $Q^r(R)$ is a Martindale right ring of quotients of R (see [13] for more on $Q^r(R)$). We say that g is *X-outer* if $\phi_g = 0$.

A subgroup G of $Aut(R)$ is called *X-outer on R* if every $1 \neq g \in G$ is X-outer.

Lemma 3.20. [18] *Let R be a semiprime ring, G is a X-outer group of ring automorphism of R.*

(1) *If R is a simple ring, then $R * G$ is a simple ring.*
(2) *If R is primitive and G is finite, then $R * G$ is primitive.*
(3) *If R is a semisimple ring and G is finite, then $R * G$ is a semisimple.*

Theorem 3.21. *Let R be a semiprime ring, G is a X-outer group of ring automorphism of R, if R is a simple ring, then $R * G$ is a Baer ring.*

Proof. Let R be a semiprime ring, G is a X-outer group of ring automorphism of R, since R be a simple ring, then $R * G$ is a simple ring by Lemma 3.20(1), since $R * G$ is a ring with identity, so $R * G$ is a semisimple ring, thus $R * G$ is a Baer ring by Lemma 3.12. □

Above Theorem proved Baerness of a skew group ring $R * G$. For fixed ring R^G, we construct a skew group ring $R^G * G$, then it is a ring under addition and multiplication of a skew group ring.

Theorem 3.22. *Let R be a reversible ring, G is a subgroup of ring automorphism of R, then skew group ring $R^G * G$ is a reversible ring.*

Proof. Suppose that $\left(\sum_{g\in G} r_g g\right)\left(\sum_{h\in G} r_h h\right) = \sum_{g\in G}\sum_{h\in G}(r_g r_h^{g^{-1}})gh = \sum_{g\in G}\sum_{h\in G}(0_{R^G})gh$,for any $\sum_{g\in G} r_g g$, $\sum_{h\in G} r_h h \in R^{\mathrm{G}} * G$, $r_g, r_h \in R^G$, then $r_g r_h^{g^{-1}} = 0_{R^G}$, $r_g r_h = 0_{R^G}$, since R is a reversible ring, then R^G is a reversible ring by Theorem 3.14, so $r_h r_g = 0_{R^G}$, $r_h r_g^{h^{-1}} = 0_{R^G}$, so $\left(\sum_{h\in G} r_h h\right)\left(\sum_{g\in G} r_g g\right) = \sum_{h\in G}\sum_{g\in G}(r_h r_g^{h^{-1}})hg = \sum_{h\in G}\sum_{g\in G}(0_{R^G})hg$. Thus $R^G * G$ is a reversible ring. □

We show that reverse of Theorem 3.22 is not hold by using following example.

Example 3.23. *Let S be a reversible ring, $G = Aut(R)$, $R = S \times S$ with addition and multiplication as follows: $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$, $(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1 b_2)$, for any $a_1, a_2, b_1, b_2 \in S$, then R is a ring, moreover R is a reversible ring, but $R^G * G$ is not a reversible.*

In fact, assume $(a_1, b_1)\cdot(a_2, b_2) = (a_1 a_2, b_1 b_2) = (0, 0)$, for any $a_1, a_2, b_1, b_2 \in S$, then $a_1 a_2 = 0$ and $b_1 b_2 = 0$, since S is a reversible ring, so $a_2 a_1 = 0$, $b_2 b_1 = 0$, then $(a_2, b_2) \cdot (a_1, b_1) = (a_2 a_1, b_2 b_1) = (0, 0)$, thus R is a reversible ring.

However, assume for any $\sum\limits_{g\in G}(a,0)g$, $\sum\limits_{h\in G}(0,b)h\in R^G*G$, $\left(\sum\limits_{g\in G}(a,0)g\right)$ $\left(\sum\limits_{h\in G}(0,b)h\right)=\sum\limits_{g\in G}\sum\limits_{h\in G}(a,0)(0,b)^{g^{-1}}gh=\sum\limits_{g\in G}\sum\limits_{h\in G}(a,0)(0,b)gh=\sum\limits_{g\in G}$ $\sum\limits_{h\in G}(0_s,0_s)gh=0$,but $\left(\sum\limits_{h\in G}(0,b)h\right)\left(\sum\limits_{g\in G}(a,0)g\right)=\sum\limits_{h\in G}\sum\limits_{g\in G}(0,b)(a,0)^{h^{-1}}$ $hg=\sum\limits_{h\in G}\sum\limits_{g\in G}(0,b)(0,a)hg,=\sum\limits_{h\in G}\sum\limits_{g\in G}(0,ba)hg$, if $ba\neq 0$, then $\sum\limits_{h\in G}\sum\limits_{g\in G}(0,ba)$ $hg\neq 0$, so R^G*G is not a reversible.

Morita Context theory is one of most important theory for research of ring. Matrix ring Morita Context ring (R,V,W,S,ψ,ϕ) is an algebraic structure which six in one.

Definition 3.24. Let R and S are rings, $V={}_RV_S$ and $W={}_SW_R$ two bimodules with bimodule map φ: $W\otimes_R V\longrightarrow S$ and map ψ: $V\otimes_S W\longrightarrow R$, given by $\varphi(w,v)\,w'=w\psi(v,w')\,,\psi(v,w)\,v'=v\varphi(w,v')\,$, for any $v,v'\in V$ and $w,w'\in W$.
Let $C=\begin{pmatrix} R & V\\ W & S\end{pmatrix}=\left\{\begin{pmatrix} r & v\\ w & s\end{pmatrix}|r\in R,s\in S,v\in V,w\in W\right\}$, Define addition of matrix for C, and multiplication as follows:
$\begin{pmatrix} r & v\\ w & s\end{pmatrix}\begin{pmatrix} r' & v'\\ w' & s'\end{pmatrix}=\begin{pmatrix} rr'+\psi(v,w') & rv'+vs'\\ wr'+sw' & \varphi(w,v')+ss'\end{pmatrix}$, Then C is a ring and called *Morita context ring.*

From reference [15] we can get condition of Morita Context ring become a Baer ring.

Theorem 3.25. [15] *Let* $C=\begin{pmatrix} R & V\\ W & S\end{pmatrix}$ *be a Morita context ring, if* $\psi=0$, $\phi=0$ *and* $Vf=V$, $We=W$ *for any* $e^2=e\in R$ *and* $f^2=f\in S$, *then* C *is a Baer ring if and only if* R *and* S *are Baer rings, and* $V=0$, $W=0$.

Theorem 3.26. [15] *Let* $C=\begin{pmatrix} R & V\\ W & S\end{pmatrix}$ *be a Morita context ring, if* $\psi=0$, $\phi=0$ *and* $Vf=V$, $We=W$ *for any* $e^2=e\in R$ *and* $f^2=f\in S$, *then* C *is a quasi-Baer ring if and only if* R *and* S *are quasi-Baer rings, and* $V=0$, $W=0$.

Theorem 3.27. [15] *Let* R *and* S *are rings,* $V={}_RV_S$ *and* $W={}_SW_R$ *are bimodules. If* $C=\begin{pmatrix} R & V\\ W & S\end{pmatrix}$ *is a Morita context ring, then* C *is a right principally quasi-Baer ring if and only if the following conditions hold:*

(1) R *and* S *are right principally quasi-Baer rings.*
(2) *For any* $a\in R$, $b\in S$, $m\in V$, $n\in W$, *there exists* $e^2=e\in R$, $f^2=f\in S$, $k\in V$, $g\in W$, *such that* $ek+kf=k$, $ge+fg=g$, $e\in r_R(nR+bW+bSg)$, $f\in r_S(aRk+aV+mS)$.

(3) *For any* $x \in R, y \in V, p \in W, q \in S$, *if* $aRx = 0$, $bSq = 0$, $nRx + bWx + bSp = 0$, $aRy + aVq + mSq = 0$, *then* $x \in eR$, $q \in fS$, $y \in eV + kS$, $p \in gR + fW$.

So we conclude that Morita Context ring have Baer property.

THEOREM 3.28. *Let* R *and* S *are rings,* $V =_R V_S$ *and* $W =_S W_R$ *are bimodules and* $\psi = 0$, $\varphi = 0$, *then the following conditions are equivalent:*

(1) R *and* S *are reversible rings.*

(2) *Morita context ring* $C = \left\{ \begin{pmatrix} r & v \\ w & s \end{pmatrix} | r \in R, s \in S, v \in V, w \in W \right\}$ *is a reversible ring.*

Proof. (1)$\Longrightarrow$(2);
If $\begin{pmatrix} r_1 & v_1 \\ w_1 & s_1 \end{pmatrix}\begin{pmatrix} r_2 & v_2 \\ w_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1r_2 + \psi(v_1, w_2) & r_1v_2 + v_1s_2 \\ w_1r_2 + s_1w_2 & \varphi(w_1, v_2) + s_1s_2 \end{pmatrix}$ $= \begin{pmatrix} r_1r_2 & r_1v_2 + v_1s_2 \\ w_1r_2 + s_1w_2 & s_1s_2 \end{pmatrix} = 0$, for any $\begin{pmatrix} r_1 & v_1 \\ w_1 & s_1 \end{pmatrix}, \begin{pmatrix} r_2 & v_2 \\ w_2 & s_2 \end{pmatrix} \in C$, then $r_1r_2 = 0$, $r_1v_2 + v_1s_2 = 0$, $w_1r_2 + s_1w_2 = 0$, $s_1s_2 = 0$. Since R and S are reversible rings, so $r_2r_1 = 0$, $s_2s_1 = 0$. Because $r_1v_2 + v_1s_2 = 0$, so $r_1v_2s_1 + v_1s_2s_1 = 0$, that is $r_1v_2s_1 = 0$, by $w_1r_2 + s_1w_2 = 0$. So $w_1r_2r_1 + s_1w_2r_1 = 0$, thus $s_1w_2r_1 = 0$. Otherwise, assume $\begin{pmatrix} r_2 & v_2 \\ w_2 & s_2 \end{pmatrix}\begin{pmatrix} r_1 & v_1 \\ w_1 & s_1 \end{pmatrix} = \begin{pmatrix} r_2r_1 + \psi(r_2, w_1) & r_2v_1 + v_2s_1 \\ w_2r_1 + s_2w_1 & \varphi(w_2, v_1) + s_2s_1 \end{pmatrix} = \begin{pmatrix} r_2r_1 & r_2v_1 + v_2s_1 \\ w_2r_1 + s_2w_1 & s_2s_1 \end{pmatrix} \neq 0$,then $r_2r_1 \neq 0$, $s_2s_1 \neq 0$, $r_2v_1 + v_2s_1 \neq 0$, $w_2r_1 + s_2w_1 \neq 0$, so $r_1r_2v_1 + r_1v_2s_1 \neq 0$, $s_1w_2r_1 + s_1s_2w_1 \neq 0$, but $r_1r_2 = 0, s_1s_2 = 0$, $s_1w_2r_1 \neq 0$ and $r_1v_2s_1 \neq 0$, contradiction.

Hence $\begin{pmatrix} r_2 & v_2 \\ w_2 & s_2 \end{pmatrix}\begin{pmatrix} r_1 & v_1 \\ w_1 & s_1 \end{pmatrix} = \begin{pmatrix} r_2r_1 & r_2v_1 + v_2s_1 \\ w_2r_1 + s_2w_1 & s_2s_1 \end{pmatrix} = 0$. thus Morita context ring $C = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ is a reversible ring.

(2)$\Longrightarrow$(1);
Since $C = \begin{pmatrix} R & V \\ W & S \end{pmatrix} = \left\{ \begin{pmatrix} r & v \\ w & s \end{pmatrix} \middle| r \in R, s \in S\ , v \in V, w \in W \right\}$ is a reversible ring.
Assume $\begin{pmatrix} r_1 & v_1 \\ w_1 & s_1 \end{pmatrix}\begin{pmatrix} r_2 & v_2 \\ w_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1r_2 + \psi(v_1, w_2) & r_1v_2 + v_1s_2 \\ w_1r_2 + s_1w_2 & \varphi(w_1, v_2) + s_1s_2 \end{pmatrix}$ $= \begin{pmatrix} r_1r_2 & r_1v_2 + v_1s_2 \\ w_1r_2 + s_1w_2 & s_1s_2 \end{pmatrix} = 0$, for any $\begin{pmatrix} r_1 & v_1 \\ w_1 & s_1 \end{pmatrix}, \begin{pmatrix} r_2 & v_2 \\ w_2 & s_2 \end{pmatrix} \in C$, then $\begin{pmatrix} r_2 & v_2 \\ w_2 & s_2 \end{pmatrix}\begin{pmatrix} r_1 & v_1 \\ w_1 & s_1 \end{pmatrix} = \begin{pmatrix} r_2r_1 & r_2v_1 + v_2s_1 \\ w_2r_1 + s_2w_1 & s_2s_1 \end{pmatrix} = 0$, that is $r_2r_1 = 0$, $s_2s_1 = 0$, since $r_1r_2 = 0$, $s_1s_2 = 0$. Hence R and S are reversible rings. □

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